3 Problems with the Duffing Equation

1) Coupling in the multiple degree of freedom system
2) Forced oscillators with a single degree of freedom ($F = A\sin(x)$)
3) Using statistical quantities in the single degree of freedom case

3.1 Coupling in Multiple Degree of Freedom Systems (Elisabeth Malsch)

Recall that last week we derived LaGrange's equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathrm{L}}{\partial q}\right) - \frac{\partial \mathrm{L}}{\partial q} = 0 \tag{3.1}$$

where $\mathrm{L} = \mathrm{T} - \mathrm{V}$ is kinetic energy less potential energy. We would like to apply this equation to the multi-dimensional Duffing oscillator, but we shall first look at the simpler model:

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}\begin{pmatrix} \ddot{x_1} \\ \ddot{x_2} \end{pmatrix} + \begin{pmatrix} k_1 - k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \tag{3.2}$$

or, more generally:

$$\mathrm{M}\ddot{\vec{x}} + \mathrm{K}\vec{x} = 0 \tag{3.3}$$

3.1.1 Linear Models with Multiple Degrees of Freedom

We wish to convert the equations (3.2, 3.3) into an eigenvalue problem of the sort $A\vec{v} = \lambda\vec{v}$, which we can solve by finding when the matrix $A - \lambda I$ is singular. The first step is to make a guess at the solution; one might reasonably suppose that $\vec{x} = e^{st}\vec{u}$ thus $\ddot{\vec{x}} = s^2 e^{st}\vec{u}$; substituting into equation 3.3 we have:

$$s^2\mathrm{M}\vec{u} + \mathrm{K}\vec{u} = 0 \tag{3.4}$$

We can then do the following two substitutions: $\lambda := s^2$ and $\vec{u} := \mathrm{M}^{\frac{1}{2}}\vec{v}$. After multiplying on the left by $\mathrm{M}^{\frac{1}{2}}$ we have:

$$\lambda\mathrm{M}^{\frac{1}{2}}\mathrm{M}\mathrm{M}^{\frac{1}{2}}\vec{v} = \mathrm{M}^{\frac{1}{2}}\mathrm{K}\mathrm{M}^{\frac{1}{2}}\vec{v} \tag{3.5}$$

Equation 3.5 is now in the desired form if we substitute $A = \mathrm{M}^{\frac{1}{2}}\mathrm{K}\mathrm{M}^{\frac{1}{2}}$ into the equation. Yet we must still compute $\mathrm{M}^{\frac{1}{2}}$, which is trivial to compute if M is diagonal, but that is not usually the case. However, Mathematica can compute the square root of a matrix M if M is symmetric, in which case one types: `MatrixPower[M, 1/2]`. If we substitute, instead, $\vec{x} = \mathrm{P}\vec{y}$ and $\ddot{\vec{x}} = \mathrm{P}\ddot{\vec{y}}$ we get:

$$\mathrm{P}\ddot{\vec{y}} + \mathrm{AP}\vec{y} = 0 \tag{3.6}$$

By left-multiplying by P^{T} we get $\ddot{\vec{y}} + \Lambda\vec{y} = 0$, and, since Λ is a diagonal matrix we have:

$$\ddot{y_i} + \lambda_i y_i = 0 \quad \forall\, i \tag{3.7}$$

3.1.2 The Duffing Model with Multiple Degrees of Freedom

We now wish to use the techniques of the previous section to find solutions for the Duffing model with multiple degrees of freedom:

$$\mathrm{M}\ddot{\vec{x}} + \mathrm{K}\vec{x} + \mathrm{K}_1\vec{x}^T\vec{x}\,\vec{x}^T = 0 \tag{3.8}$$

We can substitute in for $\vec{x} = e^{st}\vec{u}$ as we did above, but this time we are left with:

$$s^2\mathrm{M}\ddot{\vec{u}} + \mathrm{K}\vec{u} + \mathrm{K}_1 e^{2st}\vec{u}^T\vec{u}\,\vec{u}^T = 0 \tag{3.9}$$

It is immediately obvious that this equation will not succumb to an eigensystem solution, as $\vec{u}^T\vec{u}\,\vec{u}^T$ can not be simplified into a product of matrices. Thus we are left with equation 3.9 as our best option.

3.2 Simple Examples of Strange Attractors (Smita Sihag)

Now we will turn our focus to *strange attractors*. These are objects that arise as subsets of phase-space for certain dynamic systems and exhibit certain properties that merit the term *strange*.

Definition 3.1. An **attractor** is a closed set A satisfying:

1. A is invariant under the action of the dynamic system
2. A attracts on an open set of initial conditions
3. A is the smallest such closed set

There is, of course, an additional requirement that makes A into a *strange attractor*, namely the system as a whole must demonstrate a sensitive dependence on initial conditions. What this is saying is that the trajectories starting from any two nearby points must diverge exponentially fast.

So we are left with two opposing properties since trajectories on A remain bounded if A is bounded, but the trajectories must diverge exponentially.

3.2.1 The Pastry Map

To reconcile these two properties one can look at something called the "pastry map." The pastry map takes a set of initial conditions, the "dough," and stretches it followed by curving the dough over into a horseshoe shape, so that if the initial conditions are bounded the final horseshoe will be bounded as well. By repeating this process one can achieve a horseshoe with an infinite number of layers, each separated by some distance from its neighbors. (For drawings of this process see Strogatz, pp. 424-425.)

3.2.2 The Baker's Map

An alternate construction is called the "baker's map," denoted B. We define $\mathrm{B} : \mathrm{I} \times \mathrm{I} \rightarrow \mathrm{I} \times \mathrm{I}$ by:

$$\mathrm{B}(x_{n+1}, y_{n+1}) = \left\{ \begin{array}{ll} (2x_n, ay_n) & 0 \leq x_n \leq \frac{1}{2} \\ (2x_n - 1, ay_n + \frac{1}{2} & \frac{1}{2} \leq x_n \leq 1 \end{array} \right\} \tag{3.10}$$

where the parameter $a \in [0, \frac{1}{2}]$. This map stretches and flattens the initial square and then cuts the result in two and stacks the two halves in every iteration. We want to show that B has a strange attractor A for every $a \leq \frac{1}{2}$. We proceed by iterating the map and recognizing that B^n(S) has 2^n horizontal strips of height a^n, so in the limit, A=$\mathrm{B}^\infty(S) = \mathrm{C} \times \mathrm{I}$ for some Cantor set C. Further, for all $a < \frac{1}{2}$, B shrinks area and for $a = \frac{1}{2}$, B preserves area, which is associated with conservative systems.

3.3 The Hénon Map (Kareen Rozen)

We will now look at an iterated map that is a variation of the Lorentz system. For all a, $b \in \mathbb{R}$ we will define the Hénon map by:

$$\begin{aligned} x_{n+1} &= y_n + 1 - ax_n^2 \\ y_{n+1} &= bx_n \end{aligned} \tag{3.11}$$

The map takes straight lines to curves and then rotates everything about the line y=x. Further the Hénon map possesses the following properties:

1. Every iteration is invertible
2. The map contracts area at the same rate across all of $\mathbb{R}^2$
3. The map has a trapping region for certain values of a, b
4. There are trajectories that escape to infinity

3.4 The Rössler System and Forced Double-Well Oscillators (Andrew Rudman)

3.4.1 The Rössler System

The Rössler system is a three dimensional extension of the Baker's and Pastry maps, but instead of being an iterated map this is a dynamic system defined by:

$$\begin{aligned} \dot{x} &= -y - z \\ \dot{y} &= x + ay \\ \dot{z} &= b + z(x - c) \end{aligned} \tag{3.12}$$

This map stretches and folds all trajectories and then "reinjects" the trajectory for another cycle, so, in a sense, the system is an iterated map. Further, a Poincaré section of the system looks like the Pastry map.

3.4.2 Forced Double-Well Oscillators

Until now all systems have been *autonomous*, so there was no explicit dependence on time. Now, however, we will look at a forced system that explicitly depends on time. Our system will be:

$$\ddot{x} + \delta\dot{x} - x + x^3 = \mathrm{F}\cos(\omega t) \tag{3.13}$$

To study this system we will look at Poincaré sections of the plot (x(t), y(t)) where $t \in 2\pi\mathbb{Z}$, so we are looking at the periodic behavior of the system in a process called *strobing*. What we see is something that looks like the Duffing oscillator that can be viewed as a cross-section of the strange attractor for the system.