List of Figures

Henon Mapping Analysis

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Abstract

A line of initial conditions is iterated with the Henon Map:

$$T : \begin{cases} x_{i+1} &= x_i \cos\alpha - (y_i - x_i^2)\sin\alpha \\ y_{i+1} &= x_i \sin\alpha + (y_i - x_i^2)\cos\alpha. \end{cases} \tag{1}$$

All points inside the domain are kept in the iterations. Any of the points which are outside the domain are not iterated further. The domain is:

$$\begin{array}{c} -1 < x < 1 \\ -1 < y < 1 \end{array} \tag{2}$$

Different initial conditions and rotations render different results which help to illustrate the basic characteristics of the mapping including various fixed points of the mapping.

1 Henon Map

The Henon mapping (eq. 1) is a simple two dimensional mapping. It can be separated into two steps $T = T_\alpha T_\beta$. The first is a nonlinear shearing step T_α, the second is a rotation step T_β.

$$T_\alpha : \begin{cases} x_{i+1/2} &= x_i \\ y_{i+1/2} &= y_i - x_i^2 \end{cases} \tag{3}$$

$$T_\beta : \begin{cases} x_{i+1} &= x_{i+1/2}\cos\alpha - y_{i+1/2}\sin\alpha \\ y_{i+1} &= x_{i+1/2}\sin\alpha + y_{i+1/2}\cos\alpha \end{cases} \tag{4}$$

The map is area preserving, and thus describes a conservative system. The Jacobian Matrix is as follows:

$$\begin{bmatrix} \frac{dx_{i+1}}{dx} & \frac{dx_{i+1}}{dy} \\ \frac{dy_{i+1}}{dx} & \frac{dy_{i+1}}{dy} \end{bmatrix} = \begin{bmatrix} \cos\alpha + 2x\sin\alpha & -\sin\alpha \\ -2x\cos\alpha + \sin\alpha & \cos\alpha \end{bmatrix} \tag{5}$$

The Jacobian, the determinant of the Jacobian Matrix (eqn. 5), is one ($\cos^2\alpha + \sin^2\alpha = 1$).

1.1 Fixed Points

The fixed points of an iterative mapping occur when the values of x and y at the $n+1$ step of the iteration has the same values were at the n iteration.

$$\left\{ \begin{array}{c} x_0 \\ y_0 \end{array} \right\} = T\left(\left\{ \begin{array}{c} x_0 \\ y_0 \end{array} \right\}\right) = \left\{ \begin{array}{c} x_0\cos\alpha - (y_0 - x_0^2)\sin\alpha \\ y_0\sin\alpha + (y_0 - x_0^2)\cos\alpha \end{array} \right\} \tag{6}$$

The fixed points depend on the angle of rotation α. The equation is second order, so there should be two fixed points.

$$\left\{ \begin{array}{c} x_0 \\ y_0 \end{array} \right\} = \left\{ \begin{array}{c} 0 \\ 0 \end{array} \right\} \quad \text{and} \quad \left\{ \begin{array}{c} 2\tan\frac{\alpha}{2} \\ 2\tan^2\frac{\alpha}{2} \end{array} \right\} \tag{7}$$

1.2 Taylor Series Expansion about the Fixed Points

The Taylor Expansion [1] around the fixed point is:

$$\vec{x}_{n+1} = T(\vec{x}_n) = T(\vec{x}_0) + T'(\vec{x}_0)(\vec{x_n} - \vec{x}_0) + \frac{T''(\vec{x}_0)}{2}(\vec{x_n} - \vec{x}_0)^2 \ldots \tag{8}$$

1.3 The Linear Expansion

Only the linear term of the expansion is used in this case and use the definition of the fixed point (eq. 6):.

$$\vec{x}_{n+1} - \vec{x}_0 \approx T'(\vec{x}_0)(\vec{x_n} - \vec{x}_0) = \lambda(\vec{x_n} - \vec{x}_0) \tag{9}$$

Thus,the eigenvalues and eigenvectors of the Jacobian Matrix (eqn. 5) dictate the behavior of the mapping around the fixed point as long as the behavior can be characterized as linear.

[1] **Taylor's Theorem.** Suppose that a function f is analytic throughout and open disk $|z - z_0| < R_0$, centered at z_0 and with a radius R_0. Then at each point z in that disk, $f(z)$ has the series representation $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n \quad (|z - z_0| < R_0)$ where $a_n = \frac{1}{2\pi i}\frac{f^{(n)}(z_0)}{n!} \quad (n = 0, 1, 2, \ldots)$.

For taylor's expansion in the real domain $\Re$: $a_n = \frac{f^{(n)}(z_0)}{n!} \quad (n = 0, 1, 2, \ldots)$.

x_0	$\{0,0\}$	$\{2\tan^2(\alpha/2), 2\tan(\alpha/2)\}$
$[\mathbf{J}]$	$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$	$\begin{pmatrix} 4-3\cos\alpha & -\sin\alpha \\ (1-3\cos\alpha)\tan(\alpha/2) & \cos\alpha \end{pmatrix}$
λ	$e^{\pm i\alpha}$	$2-\cos\alpha \pm \sqrt{\frac{7-8\cos\alpha+\cos 2\alpha}{2}}$
$\vec{v}$	$\begin{pmatrix} \pm i \\ 1 \end{pmatrix}$	$\frac{4-4\cos\alpha \pm \sqrt{7-8\cos\alpha+\cos 2\alpha}}{(\sin\alpha - 4\cos\alpha\tan(\alpha/2))\sqrt{2}}$

Thus, the behavior around the first fixed point at $\{0,0\}$ is elliptic, while around the second it is hyperbollic.

1.4 Higher order Fixed Points

The second order mapping T^2 (where the map is applied twice) has two additional fixed points whose position also depends on α. One of the additional points is stable while the other is not.

Following this pattern one would expect T^n order mappings should result in 2^n fixed points. If these fixed points are visible in a numerical test of the mapping they form a circle of n fixed points around a lower order fixed point. (Henon 1969).

1.5 Angle of Rotation

The change in the angle of rotation α effects the behavior of the iteration around the fixed point, as well as the placement of the all the fixed points except the point at $\{0,0\}$.

The placement of the fixed points in the problem depends on α (see eq. 7). When $\alpha = \pi$ the second fixed point does not exist.

1.6 Initial Conditions

The different initial conditions fill different areas of the phase space after 100 iterations. Only the points landing inside the domain $-1 < x < 1$ and $-1 < y < 1$ are iterated further (see figure 1).

Figure 1: Domain

Figure 2: Line Initial Condition

Figure 3: Circle Initial Condition

Figure 4: Sin Initial Condition

Figure 5: The Line After 100 Iterations, $\cos \alpha = -.05$

Figure 6: The Sin After 100 Iterations, $\cos\alpha = -.05$

Figure 7: The Line After 100 Iterations, $\cos\alpha = .4$

Figure 8: The Circle After 100 Iterations, $\cos\alpha = .4$

Figure 9: The Line After 100 Iterations, $\cos\alpha = .24$

Figure 10: The Shifted Line After 100 Iterations, $\cos\alpha = .24$

Figure 11: The Circle After 100 Iterations, $\cos \alpha = .24$

Figure 12: The Sin Curve After 100 Iterations, $\cos \alpha = .24$

Figure 13: The Shifted Sin Curve After 100 Iterations, $\cos \alpha = .24$

Figure 14: The Cos Curve After 100 Iterations, $\cos\alpha = .24$

2 Observations

For each test iteration the set of points seem to approach radii about the origin. However, as the order of iterations increases the points either start circling fixed points (elliptic behavior) other than the origin or they scatter and seem to fill the area between the radii which are attached at a symmetric number of fixed points around which there is hyperbollic behavior. This is especially clear for the sine curve initial condition when $\cos\alpha = .24$ (fig. 12), although all the plots with $\cos\alpha \geq .4$ exhibit this behavior.

2.1 The Effect of the Initial Conditions

Changing initial conditions does not seem to change the placement of the fixed points, only to dictate how the points approach to or away from the points during iteration. Some Initial conditions lead to iterations that approach only some of the prominent fixed points or their radii (fig. 8, 11, 10).

Regardless of the initial conditions the areas to which the initial conditions all the example mappings appear to be symmetrical.

2.2 The Effect of Changing α

Changing α seems to change which order of transformation has more prominent fixed points. For the $\cos\alpha = -.05$ calculation there are eight fixed points, four elliptic and four hyperbollic, circle the origin ($\{0,0\}$) (fig. 6). If the initial points are too close to the lines of symmetry these orbiting fixed points no longer not seem to attract (fig. 5).

Moving away from the origin more bands governed by new fixed points begin to form. For $\cos\alpha = .4$ the twelve fixed points, six elliptic and six hyperbollic, which circle the origin are followed by twenty-six fixed points, twelve elliptic and twelve hyperbollic (fig. 7, 8).

For $\cos\alpha = .24$, there are 10 fixed points in the closest circle around the origin, five are elliptic and five hyperbollic, but outside this radius the fixed points do not seem to follow the pattern as they did for the second radius of $\cos\alpha = .4$. The bands of radii containing fixed points seem to be almost on top of one another. The pattern of fixed points seems more irregular. Nonetheless the $\cos\alpha = .24$ plots seem to approach a symmetric picture just like all the other examples before have.

2.3 Self Similarity

On the web page http://www.math.ubc.ca/people/faculty/cass/www/henon.html this henon mapping can be calculated for different α and zoomed in upon. Zooming into areas around hyperbollic fixed points shows an set of points which seem to be symmetric around the hyperbollic fixed point. In fact, initial points started on one side of the hyperbollic fixed point seem to jump back and forth over the hyperbollic point.

Also, around every higher order fixed point the behavior of the points is similar to their behavior around the origin.

3 Errors in Computation

Errors may have been introduced into the computation of these mappings. The machine epsilon, roundoff error, of the computer used to compute the maps is on the order of 10^{-12}. Also the initial conditions were discretized into $1,000$ discrete points and iterated only a hundred times. From the web page it is clear that the in between points can have very different behaviors than those of their neighbors.

4 Conclusions

The effect of higher order fixed points on the Henon Mapping can not be neglected, especially for larger distances from the origin. A linear analysis of this simple nonlinear 2-D mapping would not have shown that any other types of attractors existed besides the stable fixed point at the origin.

Detailed analysis of higher order fixed points and their relationship to α may give more information about the mapping in the specific areas around those points.

5 Reference

M. Hénon "Numerical Study of Quadratic Area-Preserving Mappings", *Quarterly of Applied Mathematics* Vol. 27-3, 10/1969. Pp. 291-312