Calculus of Variations

Consider a function with fixed endpoints $x(t_0) = x_0$ and $x(t_1) = x_1$ which is a piceswise smooth scalar function defined for all $t \in [t_0, t_1]$. There exists a scalar function of this function $x(\cdot)$, its derivative $\dot{x}(\cdot)$ and time t: $f[t, x, \dot{x}]$ which is also defined throughout the entire interval $t \in [t_0, t_1]$. It is convenient that this new function $f(\cdot)$ is continuous and contains as many partial derivatives as necessary. The **functional** $J(\cdot)$ is now the sum of $f(\cdot)$ over the range of t.

$$J(x(\cdot)) \hat{=} \int_{t_0}^{t_1} f[t, x(t), \dot{x}(t)] \mathrm{d}t \tag{1}$$

The **global absolute minimum** of $J(\cdot)$ occurs at x^* if and only if $J(x * (\cdot)) \leq J(x(\cdot)) \quad \forall x \in \{\text{domain}\}$. The **local minimum** of the integral occurs at x^* if within the immediate neighborhood of x^* the values of $J(\cdot)$ are greater than those at x^*. At a local minimum — a global minimimum is also a local minimum – the rate of change of the functional is zero, the function is **stationary**.

Using various theorems of the calculus of variations presented in the book the Euler-Lagrange form can be derived

$$f_{,x} = f_{,\dot{x}t}\dot{x}^*(t) + f_{,\dot{x}\dot{x}}\ddot{x}^*(t) \quad \forall t \in \mathbf{I} \tag{2}$$

Where $\mathbf{I}$ is the domain on which all the derivatives are continuous.

For mechanics problems the inverse function is of primary importance. If the function $x(t) = g(t, \alpha, \beta)$ is written in terms of a two paramater function, find the integrands $f(\cdot)$ which make the function $J(\cdot)$ stable. Assume that there exist continuous functions $\phi(\cdot)$ and $\psi(\cdot)$ to eliminate the constants.

$$\begin{aligned} x(t) &= g[t, \phi(t, x(t), \dot{x}(t)), \psi(t, x(t), \dot{x}(t))] \\ \dot{x}(t) &= g_t[t, \phi(t, x(t), \dot{x}(t)), \psi(t, x(t), \dot{x}(t))] \end{aligned} \tag{3}$$

Then $\ddot{x}(t) = G[t, x(t).\dot{x}(t)] = g_{tt}[t, \phi(t, x(t), \dot{x}(t)), \psi(t, x(t), \dot{x}(t))]$. And

$$f_{,x} - f_{,\dot{x}t} - \dot{x}(t) f_{,\dot{x}x} = G f_{,\dot{x}\dot{x}} \tag{4}$$

This solution must hold for every initial condition $\{x[t_0], \dot{x}[t_0]\}$.

Letting $\mathbf{M}(t, x, \dot{x}) \hat{=} f_{,\dot{x}\dot{x}}(t, x, \dot{x})$. And assuming that the derivative operation is linear $f_{,xr} = f_{,rx}$, then

$$\mathbf{M}_{,t} + \dot{x}\mathbf{M}_{,x} + G\mathbf{M}_{,\dot{x}} + G_{,\dot{x}}\mathbf{M} = 0 \tag{5}$$

The general solution to this pde is $\mathbf{M} = \frac{\mathbf{\Phi}}{\mathbf{\Theta}}$. Where $\mathbf{\Phi}$ is differentialbe nonzero but otherwise arbitrary and

$$\mathbf{\Theta} \hat{=} \exp\left\{ \int G_{,\dot{x}} \left[t, g(t, \alpha, \beta), g_t(t, \alpha, \beta)\right] \mathrm{d}t \right\} \tag{6}$$

Finally functions $f(\cdot)$ can be found by integrating

$$f(t, x, \dot{x}) = \int_0^{\dot{x}} \int_0^q \mathbf{M}(t, x, p) \mathrm{d}p \mathrm{d}q + \dot{x}\lambda(t, x) + \mu(t, x) \tag{7}$$

Where $\lambda(\cdot)$ and $\mu(\cdot)$ are otherwise arbitrary functions which satisfy the continuity conditions and are defined on the required domain.

Example

Consider a unit mass subjected to a force $P(t, x)$, where t is time and x is displacement. For rectilinear motion it follows from Newton's second law:

$$\ddot{x}(t) = P(t, x(t)) \tag{8}$$

Suppose the applied force is derivable from potential, that is there is a function $U(\cdot)$ such that

$$P(t, x) = \frac{\partial U(t, x)}{\partial x} \tag{9}$$

It follows from Hamilton's principle that the equation of motion for the particle is

$$J(x(\cdot)) = \int_{t_0}^{t_1} \left[\frac{1}{2}\dot{x}^2(t) + U(t, x(t))\right] \mathrm{d}t \tag{10}$$

And the Euler-Lagrange equation returns $\ddot{x} - U, x = 0$, the same as Newton's law of motion.

To find other stationary principles that apply to force P (other than Hamiltons) consider an equation of motion of the form:

$$\ddot{x}(t) = G\left[t, x(t), \ddot{x}(t)\right] \tag{11}$$

There are an infinity of solutions, nevertheless try the form:

$$G(t, x, \dot{x}) = a(t, x) + b(t)r + c(x)r^2. \tag{12}$$

Choose $\mathbf{\Phi}(\alpha, \beta) = 1$ for simplicity. Then

$$\mathbf{M}(t, x, r) = \mathbf{\Theta}^{-1}[t, \phi(t, x, r), \psi(t, x, r)] \tag{13}$$

Combine to find

$$\Theta\left[\cdots\right] = \exp\{\int b(t)\mathrm{d}t + 2\int c(x)\mathrm{d}x\} \tag{14}$$

Substituting into $f(\cdot)$

$$f(\cdot) = \frac{1}{2}\mathbf{\Theta}^{-1}(t, x)\dot{x}^2 + \int a(t, x)\mathbf{\Theta}^{-1}(t, x)\mathrm{d}x \tag{15}$$

In other words, the Euler-Lagrange equation for the integral is

$$\ddot{x} = a(t, x(t)) + b(t)\dot{x}(t) + c(x(t))\dot{x}^2(t) \tag{16}$$

The force function is related to the potential

$$G(t, x, \dot{x}) = a(t, x) = \frac{\partial U(t, x)}{\partial x} \tag{17}$$

And finally

$$f(t, x, r) = \frac{1}{2}r^2 + U(t, x) \tag{18}$$

Which is a special case of Hamilton's principle.