Inelasticity

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1 Basic definitions

Elasticity: The strain energy in an elastic material depends only on the strain and the temperature.

Inelasticity: The strain energy of an inelastic material depends on the internal variables including but not limited to strain and temperature.

Heredity: A material whose deformation depends on the history of loading. According to the *axiom of non-reactivity* the deformation at the present time t is due only to the forces that acted in the past, not in the future.

Viscoelasticity: A viscoelastic material is exhibits time dependent behavior under the application of stress and strain.

- **Creep:** A creep function $c(t)$ describes the elongation over time produced by a sudden application of a constant force of unit magnitude at time $t = 0$.
- **Relaxation:** A relaxation function $k(t)$ describes the force required to produce an elongation which changes from zero to unity at time $t = 0$ and remains at unity thereafter.

Bauschinger Effect: Elongation and compression in a specimen do not generate the same stresses. This is attributable to a form of residual stress occurring at grain boundaries ([5]:8).

Plasticity: Inelastic material which exhibits time independent unrecoverable deformations.

- **Yield Criteria:** The combination of primarily deviatoric stresses which causes yield in a material.
- **Flow Potential:** A scalar function of the deviatoric stresses and possibly the flow history. Its derivative in terms of stress times a positive scalar function gives the flow law.
- **Flow Law:** The rate of change of strain. It is a function of the internal variables including stress.

2 Review of field equations

Mechanics of continuum, and elastic constitutive relations govern the behavior of an elastic solid ([7]: 1–55). See appendix for more explanation. The governing equations for mechanics of solids are derived from continuum mechanics. In mechanics of solids elasticity theory is applied to solid objects. Concepts of *stress*, *deformation* and *constitutive equations* are used directly. Also, *solvability conditions* are used to prescribe the boundary conditions.

These equations:

- Strain displacement equations

$$\widetilde{\epsilon} = \frac{1}{2}(\nabla\vec{u} + \vec{u}\nabla) \tag{1}$$

- Conservation of linear momentum - Equilibrium

$$\nabla \cdot \widetilde{\sigma} + \vec{f} = \rho\ddot{\vec{u}} \tag{2}$$

- Constitutive (Linear Elastic)

$$\widetilde{\sigma} = \lambda \mathrm{tr}(\widetilde{\epsilon})\widetilde{I} + 2\mu\widetilde{\epsilon} \tag{3}$$

- Compatibility

$$\nabla \times \epsilon \times \nabla = 0 \tag{4}$$

3 Inelastic constitutive relations ([7]: 55-68)

The strain at any point in a body is not completely determined by the current stress and temperature there as it is in an elastic solid. Thus strain is a function of stress, temperature and additional internal variables $\vec{\xi}$.

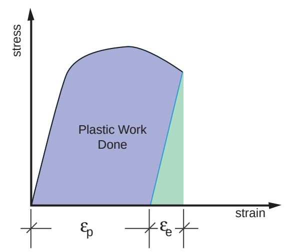
$$\widetilde{\epsilon} = \widetilde{\epsilon}(\widetilde{\sigma}, T, \vec{\xi}) \tag{5}$$

For a rate dependent body the internal variables also define the rate of evolution, or the equations of evolution

$$\dot{\xi}_\alpha = g_\alpha(\widetilde{\sigma}, T, \vec{\xi}). \tag{6}$$

Unlike for an elastic solid the relation $\widetilde{\epsilon} = \widetilde{\epsilon}(\widetilde{\sigma}, T, \vec{\xi})$ can not always be inverted to find the stress $\widetilde{\sigma} = \widetilde{\sigma}(\widetilde{\epsilon}, T, \vec{\xi})$. But, if it is possible then the equations of evolution, or rate equations can be described in terms of strain.

$$\dot{\xi}_\alpha = g_\alpha(\widetilde{\sigma}(\widetilde{\epsilon}, T, \vec{\xi}), T, \vec{\xi}) = \bar{g}_\alpha(\widetilde{\epsilon}, T, \vec{\xi}). \tag{7}$$



For inelastic bodies undergoing infinitesimal deformation it is possible to decompose elastic and inelastic strain:

$$\tilde{\epsilon} = \tilde{\epsilon}^e + \tilde{\epsilon}^i \tag{8}$$

Newtonian viscosity used in fluid mechanics is formulated using this superposition: $\tilde{\sigma} = K\mathrm{tr}(\tilde{\epsilon})\tilde{I} + 2\eta\tilde{\epsilon}^v$. Where $\tilde{\epsilon}^v$ is the viscous strain.

3.1 Models for creep and relaxation functions of Linear Viscoelastic Material [4]

Bolzmann solid viscoelastic materials retain linearity between load and deflection, but the linear relationship depends on a third parameter time. For this class of material the present state of deformation can not be determined completely unless the entire history of loading is known.

A one dimensional simple bar made of such a material behaves as follows when fixed at one end and subjected to a force in the direction of the axis at the the other end.

$$du(t) = c(t-\tau)\frac{dF}{dt}(\tau)d\tau \tag{9}$$

Here the force at time t is $F(t)$ and it is continuous and differentiable. In the small time period $d\tau$ the change in loading is $\frac{dF}{dt}d\tau$. Thus, the change in elongation of the bar $du(t)$ is proportional to the time interval $(t-T)$ and the increase in force.

If the origin of time is the beginning of motion and loading the equations above can be integrated to derive the Ludwig Bolzmann (1844-1906) constitutive equations.

$$u(t) = \int_0^t c(t-\tau)\frac{dF}{dt}(\tau)d\tau \tag{10}$$

similarly

$$F(t) = \int_0^t k(t-\tau)\frac{du}{dt}(\tau)d\tau \tag{11}$$

Vito Voltera (1860-1940) extended this formulation —coining the term heredity law— for any functional relation of the type of the Bolzmann equation.

Linear Heredity Material: A linear material whose deformation depends on the history of loading.

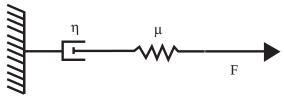
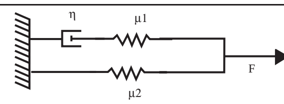
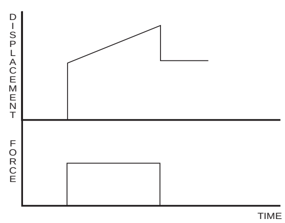
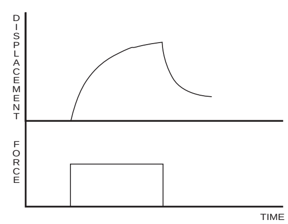
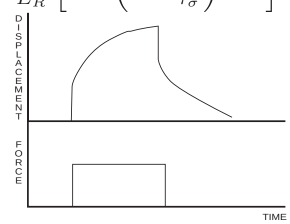
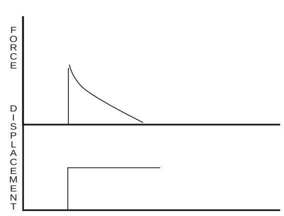
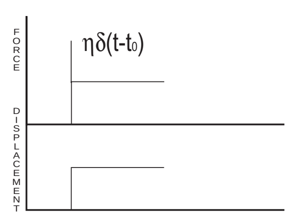
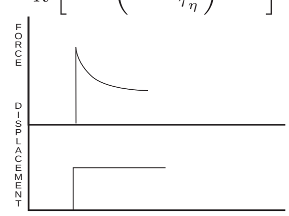
Relaxation Function: $k(t)$ the force required to produce an elongate which changes, at $t = 0$ from zero to unity and remaining at unity thereafter.

Creep Function: $c(t)$ the elongate produced by a sudden application at $t = 0$ of a constant force of unit magnitude.

Axiom of non-reactivity: The deformation at the present time t is due only to forces that acted in the past, not in the future. Thus,

$$c(t) = 0 \quad \& \quad k(t) = 0 \tag{12}$$

Viscoelasticity can be modeled with springs and dampers

	Maxwell model:	Voigt model:	Standard Linear model:
MODEL			
FORCE & DISPLACEMENT	$\dot{u} = \frac{\dot{F}}{\mu} + \frac{F}{\eta}$	$F = \mu u + \eta \dot{u}$	$F + \tau_\eta \dot{F} = E_R(u + \tau_\sigma \dot{u})$
INITIAL CONDITION	$u(0) = \frac{F(0)}{\mu}$	$u(0) = 0$	$\tau_\eta F(0) = E_R \tau_\sigma u(0)$
CREEP	$c(t) = \left(\frac{1}{\mu} + \frac{1}{\eta}\right) = \mathbf{1}(t)$ 	$c(t) = \frac{1}{\mu}(1 - e^{-(\frac{\mu}{\eta})t})\mathbf{1}(t)$ 	$c(t) = \frac{1}{E_R}\left[1 - \left(1 - \frac{\tau_\eta}{\tau_\sigma}\right) e^{\frac{-t}{t_\sigma}}\right] \mathbf{1}(t)$ 
RELAXATION	$k(t) = \mu e^{-(\frac{\mu}{\eta})t}\mathbf{1}(t)$ 	$k(t) = \eta\delta(t) + \mu\mathbf{1}(t)$ 	$k(t) = E_R\left[1 - \left(1 - \frac{\tau_\sigma}{\tau_\eta}\right) e^{\frac{-t}{t_\eta}}\right] \mathbf{1}(t)$ 

Where $\delta(t)$ is the Dirac-delta function, and $\mathbf{1}(t)$ is the unit step function and τ_η and τ_σ are constants.

3.2 Sinusoidal Oscillations in a Viscoelastic Material [4]

Assume a body is forced to perform simple harmonic oscillations. To find the steady state response assume that the forcing function has been acting on the body for an indefinitely longtime and that all initial transient disturbances have died out. Substitute $t - \tau = \xi$ into equation (1.14).

$$u(t) = \int_0^\infty c(\xi) \frac{dF}{dt}(t - \xi) d\xi \tag{13}$$

This equation is true for any Bolzmann material excited with any function $F(x)$. When the forcing function is a simple harmonic oscillation the following equation is true $F(t) = F_0 e^{iwt}$.

$$\begin{aligned} u(t) &= \int_0^\infty c(\xi) iw F_0 e^{iw(t-\tau)} d\xi \\ &= iw F_o e^{iwt} \int_0^\infty c(\xi) e^{-iwt} d\tau \end{aligned} \tag{14}$$

Since c(t)=0 when t¡0, replace the lower limit of the integral by $-\infty$ and writhe the integral in the conventional form of the fourier transformation.

$$\bar{c} = \int_{-\infty}^\infty c(\tau) e^{-iw\tau} d\tau \tag{15}$$

Assuming that the fourier integral exists,

$$u(t) = iw F_0 \bar{c}(w) e^{iw\tau} = u_0 e^{iw\tau} \tag{16}$$

Thus, the ratio u_0/F_0 is a complex number and may be written as

$$\frac{u_0}{F_0} = \frac{1}{\mathcal{M}} = iw\bar{c}(w) = |w\bar{c}(w)| e^{-i\delta}. \tag{17}$$

Where $\mathcal{M}$ is called the complex modulus of viscoelastic material. The angle δ is the phase angle by which the strain lags the stress. the tangent of δ is used to measure the internal friction of a linear viscoelastic material.

$$\tan\delta = \frac{imaginary\ part\ of\ \mathcal{M}}{real\ part\ of\ \mathcal{M}} \tag{18}$$

also,

$$\frac{F_0}{u_0} = \mathcal{M} = iw\bar{k}(w) \tag{19}$$

where $\bar{k}(w)$ is the Fourier transform of the relaxation function

$$\bar{k} = \int_{-\infty}^\infty k(\tau) e^{-iw\tau} d\tau. \tag{20}$$

the relation between $\bar{c}(w)$ and $\bar{k}(w)$ is

$$-w^2 \bar{c}(w) \bar{k}(w) = 1. \tag{21}$$

3.3 Structural Problems of Viscoelastic Materials [4]

Generalized Hook load-displacement law using hereditary law.

$$u_i = \sum_{j=1}^{n} \int_0^t C_{ij}(t-\tau)\frac{d}{d\tau}F_j(\tau)d\tau, \qquad i = 1, \ldots, n. \tag{22}$$

$C_{ij}(t)$ is the creep function, the deflection $u_i(t)$ produced by a unit step function $F_j(t) = \mathbf{1}(t)$ acting at the point j. These creep functions are not always easy to deduce or measure.

3.3.1 Multiaxial behavior of viscoelastic material

The creep and relaxation functions can be formulated as fourth order tensors. If the material is isotropic then only two constants are needed to describe the behavior. Notice that internal variables are not required in this representation.

3.4 Flow rule from flow potential

It is possible in general to define a flow law or rate equation for $\widetilde{\epsilon}^i$ by assuming that $\epsilon^i = \epsilon^i(\xi)$ and applying the chain rule.

3.4.1 Generalized Potential and Generalized Normality ([7]:66)

Assume that the rate equations can be defined in terms of a potential Ω which depends only on thermodynamic forces $\vec{p}$

$$\dot{\vec{\xi}} = \frac{\partial \Omega}{\partial \vec{p}} \tag{23}$$

Using the Gibbs function, or the complementary free energy -density $\chi = \rho^{-1}\widetilde{\sigma} : \widetilde{\epsilon} - \psi$, where ψ is the Helmholz free energy. The thermodynamic forces are

$$\vec{p} = \rho\frac{\partial \chi}{\partial \vec{\xi}} \quad \text{and} \quad \widetilde{\epsilon} = \rho\frac{\partial \chi}{\partial \widetilde{\sigma}} \tag{24}$$

Now using the cain rule find:

$$\dot{\widetilde{\epsilon}}^j = \frac{\partial \widetilde{\epsilon}}{\partial \vec{\xi}} \cdot \dot{\vec{\xi}} = \frac{\partial \vec{p}}{\partial \widetilde{\sigma}}\dot{\vec{\xi}} \tag{25}$$

Using the generalized potential find

$$\dot{\widetilde{\epsilon}}^j = \frac{\partial \Omega}{\partial \widetilde{\sigma}} \tag{26}$$

A sufficient condition fo the existence of a generalized potential was found by Rice in 1971. The condition is that each of the rate functions depend on the stress only through its own conjugate thermodynamic force $\vec{p}$. It is usually mathematically convenient to describe Ω as a convex function of $\vec{p}$. thus for any $\vec{p}^*$ such that $\Omega(\vec{p}^*) \leq \Omega(\vec{p})$, $(p - p^*) \cdot \dot{\vec{\xi}} \geq 0$.

3.4.2 Complete stress strain relations ([5])

Given a yield criteria in the form

$$f(J_2', J_3') = c \tag{27}$$

where f does not depend on strain history.

$$\mathrm{d}f = \frac{\partial f}{\sigma_{ij}}\mathrm{d}\sigma_{ij} = \frac{\partial f}{\partial J_2'}\mathrm{d}J_2' + \frac{\partial f}{\partial J_3'}\mathrm{d}J_3' \tag{28}$$

To assure that $\mathrm{d}\epsilon_{ij}^P$ is zero for a neutral change in stress assume

$$\mathrm{d}\epsilon_{ij}^p = G_{ij}\mathrm{d}f \tag{29}$$

where G_{ij} is a symmetric tensor. $G_{ii} = 0$ since hydrostatic stress does not produce plastic deformation. Thus G_{ij} can be written in potential form

$$G_{ij} = h\frac{\partial g}{\partial \sigma_{ij}} \tag{30}$$

Where h and g are scalar functions of the deviatoric invariants and possibly of the strain-history.

$$\mathrm{d}\epsilon_{ij}^p = h\frac{\partial g}{\partial \sigma_{ij}}\mathrm{d}f. \tag{31}$$

Flow Potential: g is a scalar function of the deviatoric stresses and possibly the flow history. It s derivative times a positive scalar function h is the flow law. h may also depend on stress and flow history.

Flow Law: the rate of change of strain. It is a function of the internal variables including stress.

3.5 Yield Criteria, Flow Rules and Hardening Rules ([7]:125-140)

It is convenient to write the stress in terms of a deviatoric stress.

$$\widetilde{S} = \widetilde{\sigma} - \frac{\mathrm{tr}(\widetilde{\sigma})}{\mathrm{tr}(\widetilde{I})}\widetilde{I} \tag{32}$$

Also for any yield function $f(\widetilde{\sigma}, \widetilde{\xi}) = \bar{f}(\widetilde{s}, \mathrm{tr}(\widetilde{\sigma}), \widetilde{\xi})$:

$$\frac{\partial f}{\partial \widetilde{\sigma}} = \frac{\partial \bar{f}}{\partial \widetilde{s}}\frac{\partial \widetilde{s}}{\partial \widetilde{\sigma}} + \frac{\partial \bar{f}}{\partial \mathrm{tr}(\widetilde{\sigma})}\frac{\partial \mathrm{tr}(\widetilde{\sigma})}{\partial \widetilde{\sigma}} = (\widetilde{\bar{f}} - \frac{1}{3}\mathrm{tr}(\widetilde{\bar{f}})\widetilde{I}) + \frac{\partial f}{\mathrm{tr}(\widetilde{\sigma})}\widetilde{I} \tag{33}$$

where $\widetilde{f} = \frac{\partial f}{\partial \widetilde{s}}$.

3.5.1 Initially Isotropic yield Criteria

The yield function f must depend only on the stress invariants of $\widetilde{\sigma}$, provided that the dependence is symmetric.

3.5.2 Yield Condition ([1])

Thus, a yield condition for a three dimensional solid can be formulated in terms of maximum shear. Using Mohr's circle or other tensor relations the maximum shear is given as $\frac{1}{2}|(\sigma_I - \sigma_{II})|$, $\frac{1}{2}|(\sigma_{II} - \sigma_{III})|$, or $\frac{1}{2}|(\sigma_{III} - \sigma_I)|$ where σ_I, σ_2 and σ_3 are the principle stresses. The actual yield condition depends on which of the given shears are negative. This is the Tresca Yield Condition.

Deformation energy may also be used to describe the yield condition. In the Mises-Hencky criterion

> Plastic flow will occur when the distortion-energy density in the material reaches the value corresponding to the yielding of a simple tensile specimen ([1],213).

The total strain energy density is $U = \frac{1}{2}\widetilde{\sigma} : \widetilde{\epsilon}$. For a linear elastic material $\widetilde{\sigma} = \lambda \mathrm{tr}(\widetilde{\epsilon}) + 2\mu\widetilde{\epsilon}$. The deformation energy is related to the deviatoric component of stress and is $U^* = \frac{1}{2G}(\widetilde{\sigma} : \widetilde{\sigma} - \frac{1}{3}(\mathrm{tr}\widetilde{\sigma})^2)$. For a uniaxial bar $U^* = \frac{(\sigma_0)^2}{6G}$. Now the Mieses-Hencky condition is formulated in terms of the principle components of stress:

$$\frac{1}{12G}\left[(\sigma_I - \sigma_{II})^2 + (\sigma_{II} - \sigma_{III})^2 + (\sigma_{III} - \sigma_I)^2\right] = \frac{(\sigma_0)^2}{6G}. \tag{34}$$

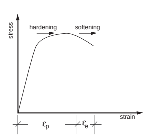
Experiments have shown the Mises condition to be more closely correlated to experimental results.

In general the yield condition of a isotropic body is a function of the second J_2 and third J_3invariants of the deviatoric components stress. The first invariant of stress is related to hydrostatic conditions. Thus in general:

$$f(J_2, J_3) = 0. \tag{35}$$

To formulate a plastic stress-strain relation it is convenient to separate the elastic and plastic components of strain $\widetilde{\epsilon} = \widetilde{\epsilon}^e + \widetilde{\epsilon}^p$. Note the the plastic component is assumed to be unchanged during plastic deformation so $\mathrm{tr}(\epsilon^p) = 0$.

3.6 Hardening Rules



Hardening Rules: a specification of the dependence of the yield criterion on the internal variables, along with the rate equations of these variables.

3.6.1 Isotropic hardening [7]

Yield functions can be reduced to the form:

$$f(\widetilde{\sigma}, \xi) = F(\widetilde{\sigma}) - k(\vec{\xi}) \tag{36}$$

Since only the yield stress is a function of the internal variables. The function $\widetilde{h}$ corresponds to the internal variables $\vec{\xi}$. So the plastic modulus H can be defined as

$$H = \begin{cases} k'(W_p)\sigma_{ij}h_{ij} \\ k'(\bar{\epsilon}^p)\sqrt{\frac{2}{3}h_{ij}h_{ij}} \end{cases} \tag{37}$$

4 Rate-independent plasticity ([7]: 112–140)

A perfectly plastic material is neither a function of time nor material imperfections. Thus it does not creep, relax, or exhibit Bauschinger's effect. The behavior of such a material depends on a yield function and its corresponding yield surface. Any change in the strain after yielding is a function only of hardening or softening of the material. The hardening is related to the plastic potential.

4.1 The Ideal Plastic Body

- Time Independent — No creep or relaxation
- No non-uniformity in the microscale and resulting differential hardening (Bauschinger Effect)
- no side effects to plastic behavior

Yield Criteria: A law which defines the limit of elasticity under any possible combination of stress components. *Assume* for a plastic body yielding depends only on the deviatoric components of stress.

$$\widetilde{\sigma}' = \widetilde{\sigma} - \frac{\mathrm{tr}(\widetilde{\sigma})}{\mathrm{tr}(\widetilde{I})}\widetilde{I} \tag{38}$$

For an isotropic material which depends only on the invariants of the deviatoric stress tensor, the yield criteria depends only on $J_2' = \frac{1}{2}\widetilde{\sigma}' : \widetilde{\sigma}'$ and $J_3' = \frac{1}{2}\widetilde{\sigma}' : (\widetilde{\sigma}' \cdot \widetilde{\sigma}')$. To allow for symmetry the yield function must be an even function of J_3'

4.2 Introduction to Plasticity ([1]: 206–280)

The stress-strain relation under uniaxial loading condition can be found experimentally. For example the behavior of a thin specimen of steel may be similar to figure 1. Notice when yielding is reached σ_y the behavior changes, the relation

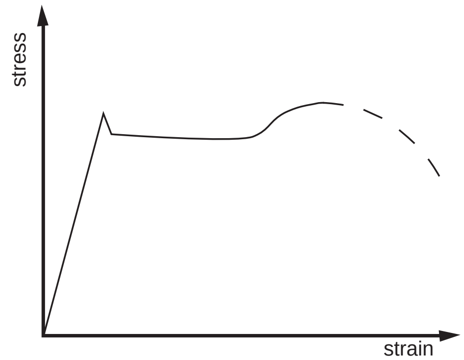


Figure 1: Possible Stress Strain relation for Steel

between stress and strain does not follow $\sigma = E\epsilon$ as for the purely linear case. One representation constructed by matching experimental stress-strain curves is the *Ramberg-Osgood* Relation

$$\epsilon = \frac{\sigma}{E} + \left(\frac{\sigma}{B}\right)^n. \tag{39}$$

For a perfectly plastic material $\sigma = \sigma_y$. Observations of experiments seem to show that plastic deformation is related only to shear stresses and not to hydrostatic pressure.

4.3 Plastic Potential

The plastic potential $g(\sigma_{ij})$ defines the ratios of components of plastic strain increments. If it is equal to $f(\sigma_{ij})$, the function which defines the yield locus. In this condition the constant contours of the plastic potential define the yield locus.

$$\mathrm{d}\epsilon^P_{ij} = h\frac{\partial f}{\partial \sigma_{ij}}\mathrm{d}f \tag{40}$$

Note that f must be independent of hydrostatic pressure if the plastic volume change is zero.

$$\frac{\partial f}{\partial \sigma_{ii}} = 0 \tag{41}$$

Also, if the function f is an even function — there is no Bauschinger effect, then the reversal of the sign of the stress reverses the sign of the stress increment. This the case for the Lévy-Mises or Reuss equations where $f = g = J_2' = \frac{1}{2}\sigma'_{ij}\sigma'_{ij}$ and $\frac{\partial f}{\partial \sigma_{ij}} = \sigma'_{ij}$.

Such an even, independent function f can be used inversely to find a unique plastic state of stress arising from a given plastic strain increment $\mathrm{d}\epsilon^P_{ij}$. Let σ^*_{ij} be any other plastic state of stress

$$f(\sigma^*_{ij}) = f(\sigma_{ij}) = c \tag{42}$$

The work done by this $\mathrm{d}\epsilon_{ij}^P$ in the strain $\mathrm{d}\epsilon_{ij}^P$ is $\mathrm{d}W_p^* = \sigma_{ij}^* \mathrm{d}\epsilon_{ij}^P$. The stationary value for varying plastic states σ_{ij}^* is when

$$\frac{\partial}{\partial \sigma_{ij}^*}(\sigma_{ij}^* \mathrm{d}\epsilon_{ij}^P - f(\sigma_{ij}^*)\mathrm{d}\lambda) = 0 \tag{43}$$

where, using Lagrange's method, a constant multiplier $\mathrm{d}\lambda$ has been added. Thus:

$$\mathrm{d}\epsilon_{ij}^P = \mathrm{d}\lambda \frac{\partial f(\sigma_{ij}^*)}{\partial \sigma_{ij}^*}. \tag{44}$$

And $\mathrm{d}\lambda = h\mathrm{d}f$.

4.4 One-dimensional plastic waves ([7]: 409–420)

If the propagation of longitudinal stress strain waves can be represented by a one-dimensional model. The inertia terms can be ignored, since the wave propagation problem is analogous to a creeping fluid problem where using scaling and Reynolds number it can be shown that the inertia term is relatively small compared with the viscous effects or shear stress. This assumption is not good near the ends of the bar.

Given x a Lagrangian coordinate along the $x-$axis, the corresponding small displacement strain ϵ and velocity v are:

$$\epsilon = \frac{\partial u}{\partial x} \quad \text{and} \quad v = \frac{\partial u}{\partial t} \tag{45}$$

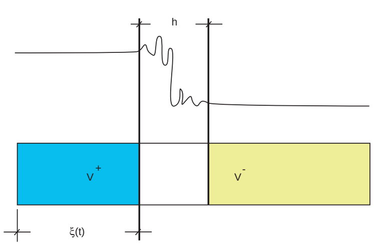
According to the kinematic compatibility relation:

$$\frac{\partial \epsilon}{\partial t} = \frac{\partial v}{\partial x}. \tag{46}$$

The equations of motion in the absence of body force reduce to:

$$\frac{\partial \sigma}{\partial x} = \rho \frac{\partial v}{\partial t} \tag{47}$$

Where σ is the nominal uniaxial stress and ρ is the mass density in the undeformed state. This reasoning can also be applied to the torsional problem.



If the front is moving at a finite speed c in the positive $x-$ direction $c = \alpha'(t) > 0$ then the jump in the velocity v can be defined as in terms of the jump, or approximated with respect to a shock thickness h and the partial derivatives of the velocity v:

$$[v] = v^- - v^+ \quad [v] \approx \pm h\frac{\partial v}{\partial x} \quad [v] \approx \pm \frac{h}{c}\frac{\partial v}{\partial t}. \tag{48}$$

Stress and strain jumps at the shock front can be defined in terms of these relations:

$$[\epsilon] = \pm\frac{1}{c}[v], \quad [\sigma] = \pm\rho c[v] \quad \text{and} \quad \rho c^2 = \frac{[\sigma]}{[\epsilon]} \tag{49}$$

Notice that the final ratio depends only on the density and the wave speed not the velocity or the shock width.

Shock Front: occurs at a point $x = \alpha(t)$ on a bar the velocity v is discontinuous.

Shock-speed equation: The ratio between the jump in stress and the jump in strain.

$$\rho c^2 = \frac{[\sigma]}{[\epsilon]} \tag{50}$$

5 Viscoelastic models ([2]: 9–14)

According to the hypothesis of fading memory the creep or relaxation effect decrease over time. Thus a the fourth-order relaxation function $\widetilde{\widetilde{G}}(t)$ or a fourth-order creep function $\widetilde{\widetilde{J}}(t)$ are continuously decreasing as a function of time.

$$\left|\frac{\mathrm{d}\widetilde{\widetilde{G}}(t)}{\mathrm{d}t}\right|_{t=t_1} \leq \left|\frac{\mathrm{d}\widetilde{\widetilde{G}}(t)}{\mathrm{d}t}\right|_{t=t_2} \quad \text{and} \quad \left|\frac{\mathrm{d}\widetilde{\widetilde{J}}(t)}{\mathrm{d}t}\right|_{t=t_1} \leq \left|\frac{\mathrm{d}\widetilde{\widetilde{J}}(t)}{\mathrm{d}t}\right|_{t=t_2} \quad \text{for } t_1 > t_2 > 0 \tag{51}$$

The difference between a *viscoelastic solid* and a *viscoelastic fluid* stated physically are: When a viscoelastic fluid is subjected to a fixed simple shear state of stress it responds with a steady state flow after the transient effects have died out, also when a fluid is subjected to a fixed simple shear state of deformation the shear stress state will eventually decay to zero: $\lim_{t\to\infty} G_1(t) = 0$. When a viscoelastic solid is subjected to a simple shear state of deformation, it will have a component of stress which remains nonzero as long as the state of deformation is maintained: $\lim_{t\to\infty} G_1(t) = C$ where C is a nonzero constant.

Subject a viscoelastic material to a simple shear state of deformation specified by the displacement components from the fixed reference configuration as:

$$u_1(\vec{x}, t) = \hat{u} X_2 h(t) \qquad u_2 = u_3 = 0 \tag{52}$$

where $h(t)$ is the unit step function. Using the infinitesimal strain displacement relations

$$\widetilde{\epsilon} = \frac{1}{2}\left(\nabla_{\vec{X}}\vec{u} + \vec{u}\nabla_{\vec{X}}\right). \tag{53}$$

The only nonzero relation between the components of stress and strain in a simple shear in cartesian coordinates is

$$s_{12}(t) = \left[\frac{G_1(t)}{2}\right]\hat{u} \tag{54}$$

where $G_{ijkl}(t) = \frac{1}{3}[G_2(t) - G_1(t)]\delta_{ij}\delta_{kl} + \frac{1}{2}[G_1(t)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})]$. In an isotropic material the fourth order tensor is defined by two independent constants. Deviatoric stress in general is $\widetilde{s} = \int_{-\infty}^{t} G_1(t-\tau)\frac{\mathrm{d}\widetilde{e}}{\mathrm{d}\tau}\mathrm{d}\tau$, where $\widetilde{e}$ is the deviatoric component of strain and strain $\widetilde{\sigma} = \int_{-\infty}^{t} G_2(t-\tau)\frac{\mathrm{d}\widetilde{\epsilon}}{\mathrm{d}\tau}\mathrm{d}\tau$. Similar relations are formed using the creep functions. The relaxation function $G_1(t) = 0$ for $t < 0$.

5.1 Harmonic response ([3])

In steady-state harmonic oscillations every forcing and response function is of the form

$$g(\vec{x}, t) = \bar{g}(\vec{x}, w)\mathrm{e}^{iwt}. \tag{55}$$

In general $\bar{g}$ can be a complex function. Correspondingly, for an isotropic viscoelastic body the Lamé constants are also frequency dependent $\lambda(w)$ and $\mu(w)$. If the viscoelastic characteristics of material can be assumed to be identical in bulk and shear then, the ratio of the frequency dependent constants is not a function of frequency, The bulk modulus is not a function of frequency:

$$\frac{\lambda(w)}{\mu(w)} = K = \frac{2\nu}{(1-2\nu)}. \tag{56}$$

For a Voigt Solid the energy loss per cycle of harmonic vibration is proportional to the excitation frequency, in a constant hysteretic solid the energy loss is independent of the frequency. The damping in a hysteretic solid can then be uncoupled in a multi degree of freedom system, as for structural or rate independent linear damping. The dynamic stiffness influence coefficients for Voigt and constant hysteretic solids can be found using the frequency dependent constitutive descriptions.

6 Viscoelastic correspondence principle ([2]: 187–221)

If a frequency response function of an elastic system is known, then the corresponding viscoelastic frequency response may be obtained from it directly.

Wave propagation in viscoelastic material on semi-infinite and infinite domains can be solved by integral transform, finite domain problems are far more complicated. Applying the findings from the infinite domains to large finite domains is usefully for practical problems.

6.1 Isothermal Wave Propagation

As for one-dimensional plastic waves, the stress-strain behavior about the strain discontinuity or shock front is used to solve the wave problem. As for the one-dimensional problem, the jump properties can be defined in terms of a jump width h, and the conservation of linear momentum can be applied across the jump.

The uniaxial viscoelastic constitutive relation is

$$\sigma(x,t) = \int_{-\infty}^{t} E(t-\tau)\frac{\partial \epsilon(x,\tau)}{\partial \tau}\mathrm{d}\tau. \tag{57}$$

Integrate by parts and substitute into the conservation of linear momentum

$$\left[E(0)\frac{\partial u(x,t)}{\partial x} + \int_{-\infty}^{t}\frac{\partial}{\partial t}E(t-\tau)\frac{\partial u(x,\tau)}{\partial x}\mathrm{d}\tau\right] = -\rho v\left[\frac{\partial u(x,t)}{\partial t}\right] \tag{58}$$

Applying the strain displacement relation and assuming $\frac{\mathrm{d}E(t)}{\mathrm{d}t}$ is bounded and continuous on $0 \leq t \leq \infty$ then the integral term contributes nothing to the jump and, as for plastic waves:

$$E(0)\left[\frac{\partial u(x,t)}{\partial x}\right] = -\rho v\left[\frac{\partial u(x,t)}{\partial t}\right] \tag{59}$$

Using kinematic compatibility as before the speed of propagation is given by $v = [E(0)/\rho]^{\frac{1}{2}}$ as in equation 49.

The result can also be found using the Laplace Transform.

$$\frac{\partial \sigma(x,t)}{\partial x} = \rho\frac{\partial^2 u(x,t)}{\partial t^2} \quad \text{using the transform} \quad sE(s)\frac{\partial \bar{\epsilon}(x,s)}{\partial x} = \rho s^2 \bar{u}(x,s) \tag{60}$$

The rod is initially assumed to be at rest

$$\frac{\partial^2 \bar{u}}{\partial x^2} - \frac{\rho s}{\bar{E}(s)}\bar{u} = 0 \tag{61}$$

Taking the transform of the viscoelastic strain displacement relation

$$\frac{\partial^2 \bar{\sigma}}{\partial x^2} - \frac{\rho s}{\bar{E}(s)}\bar{\sigma} = 0 \tag{62}$$

The same equation results from the laplace transform of strain, noticing that the particle velocity $\bar{v} = s\bar{u}$. The general solution of the equations is given by

$$(\bar{\sigma}, \bar{\epsilon}, \bar{u}, \bar{v}) = A(s)\mathrm{e}^{\Omega(s)x} + B(s)\mathrm{e}^{-\Omega(s)x} \quad \text{where} \quad \Omega(s) = \left[\rho s/\bar{E}(s)\right]^{\frac{1}{2}} \tag{63}$$

Here $A(s)$ and $B(s)$ are determined by the boundary conditions.

6.2 Reflection of Harmonic Waves

The fourier transformed equation of motion for an isotropic solid is is:

$$\mu^*(iw)\nabla^2\bar{\mathbf{u}} + [\lambda^*(iw) + \mu^*(iw)]\nabla(\nabla \cdot \bar{\mathbf{u}}) = -\rho w^2 \bar{\mathbf{u}}. \tag{64}$$

These equations are uncouples since the inertial terms can be neglected. The resulting equations of motions are

$$\mu^*\nabla^2\bar{\mathbf{u}} = -\rho w^2\bar{\mathbf{u}} \quad \text{and} \quad (\lambda^* + 2\mu^*)\nabla^2\bar{\mathbf{u}} = -\rho w^2\bar{\mathbf{u}}. \tag{65}$$

These equations govern the propagation of shear S and irrotational P waves respectively.

7 Viscoplasticity ([7]:102–110)

7.1 Yield surface

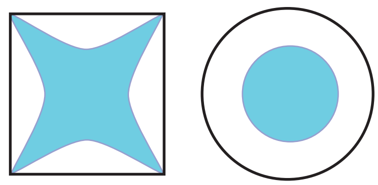


Figure 2: Yield Surfaces for Different Cross-Sections in Torsion

The yield surface separates regions where inelastic strain-rate tensor is zero and where it is not zero. In terms of yield function $f(\cdot)$, the yield surface is defined by $f(\widetilde{\sigma}, T, \vec{\xi}) = 0$.

7.1.1 Drucker's Postulate

> If a unit volume of an elastic-plastic specimen under uniaxial stress is initially at stress $\widetilde{\sigma}$ and plastic strain at $\widetilde{\epsilon}^P$ and an "external agency" slowly applies an incremental load resulting in a stress increment $\mathrm{d}\widetilde{\sigma}$ and subsequently removes it then $\mathrm{d}\widetilde{\sigma}\mathrm{d}\widetilde{\epsilon} = \mathrm{d}\sigma(\mathrm{d}\widetilde{\epsilon}^e\mathrm{d}\widetilde{\epsilon}^P)$ is the work performed by the external agency in the course of incremental loading and $\mathrm{d}\widetilde{\sigma}\mathrm{d}\epsilon^P$ is the work performed in the course of the cycle consisting of the application and removal of the incremental stress.

A stable or work-hardened material is one in which the work done in an incremental loading is positive and the loading- unloading cycle is non-negative. The plastic strain increment $\mathrm{d}W_p$ is essentially positive since plastic distortion is an irreversible process in the thermodynamic sense. In general

$$\mathrm{d}\widetilde{\sigma} : \mathrm{d}\widetilde{\epsilon} > 0 \quad \text{and} \quad \mathrm{d}\widetilde{\sigma} : \mathrm{d}\widetilde{\epsilon}^P \geq 0 \quad \text{and} \quad W_p = \int \sigma_{ij}\mathrm{d}\epsilon_{ij}^P. \tag{66}$$

For both work hardened and perfectly plastic material Drucker's Inequality holds

$$\dot{\sigma} : \dot{\epsilon}^P \geq 0 \tag{67}$$

The plastic strain rate can not act opposite to the stress rate. In general for a complete loading $\widetilde{\sigma}$ and unloading cycle $\widetilde{\sigma}^*$:

$$(\widetilde{\sigma} - \widetilde{\sigma}^*)\dot{\epsilon}^P \geq 0 \tag{68}$$

This is also know as the postulate of maximum plastic dissipation ([7],118).

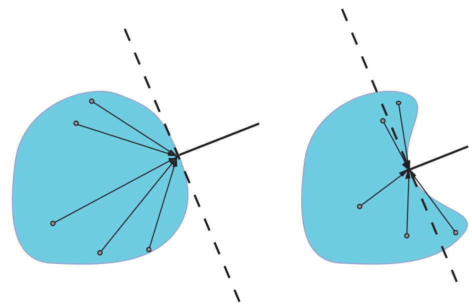


Figure 3: Drucker's Postulate and concavity

7.1.2 Consequences of the Maximum-Dissipation Postulate

If a yield surface is everywhere smooth, that is a well-defined tangent plane and normal direction exist at every point. For the inequality to be valid for all stresses inside the tangent line the rate of plastic strain must be directed along the outward normal associated with the tangent point. If there are any stresses lying on the outward side of the tangent the inequality is violated, that is the yield surface in stress space is convex.

8 Applications

8.1 Column Buckling

The buckling curve predicted for an elastic column is never reproducible experimentally. Consider instead that the material is plastic.

$$\begin{array}{lll} \text{if } \dot{\epsilon} > 0 & \text{then} & \dot{\sigma} = E_t\dot{\epsilon} \\ \text{if } \dot{\epsilon} < 0 & \text{then} & \dot{\sigma} = E\dot{\epsilon} \end{array} \tag{69}$$

In the plastic range, if the deformation is increasing the material hardens, if the deformation is decreasing it does not.

Next, a deformed shape must be assumed. Shanley's theory assumes that $\dot{\epsilon} > 0$ for all material in the column, that is the work done in compression is significantly greater than that done in tension if a column begins to deform.

The strain in a column from is $\epsilon = \epsilon_0 + k\xi$. Where k is the curvature $\frac{d^2w}{dx^2}$, w is the lateral displacement, and x is the axial axis. ϵ_0 is average strain, and ξ is measured from the center of area of the normal cross section.

$$\dot{\epsilon} = \dot{\epsilon}_0 + \dot{k}\xi \tag{70}$$

Since the strain is always increasing

$$\dot{\sigma} = E_t\dot{\epsilon} = E_t(\dot{\epsilon}_0 + \dot{k}\xi) \tag{71}$$

The compressive force $\dot{P}$ then is

$$\dot{P} = \int_A \dot{\sigma} \mathrm{d}A = \int_A E_t(\dot{\epsilon}_0 + \dot{k}\xi)\mathrm{d}A = E_t\dot{\epsilon}_0 A + E_t\dot{k}\int \xi \mathrm{d}A \tag{72}$$

Since ξ is located at the center of gravity $\dot{P} = AE_t\dot{\epsilon}_0$. And the resulting buckling load is $P = \frac{\pi^2 E_t(\sigma) I}{L^2}$.

If a material is perfectly plastic only the elastic core resists the load. Thus, the location of the yield surface defines the behavior of the column.

8.2 Torsion

Example: Thin Walled Tube

One simple theory for plastic flow which is consistent with the Mises -Hencky yield condition is that the the ratio of increment of each plastic strain component to its corresponding deviatoric stress component remains constant thus:

$$\frac{\mathrm{d}\widetilde{\epsilon}^p}{\hat{\widetilde{\sigma}}} = \mathrm{d}\lambda. \tag{73}$$

Where $\hat{\widetilde{\sigma}}$ is the deviatoric component of strain.

Consider a thin walled tube with zero initial strain under torsion and axial stress. Now the state of strain in the bar is $\widetilde{\sigma} = \left.\begin{pmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{21} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right|_{\hat{\imath}:\text{axial},\hat{\jmath}:\text{radius}}$. The twisting angle is held constant while tensile strain is applied. In general an increment in strain is:

$$\mathrm{d}\epsilon_{12} = \mathrm{d}\epsilon_{12}^e + \mathrm{d}\epsilon_{12}^p \quad \text{and} \quad \mathrm{d}\epsilon_{11} = \mathrm{d}\epsilon_{11}^e + \mathrm{d}\epsilon_{11}^p. \tag{74}$$

Substitute using the constitutive relations:

$$\mathrm{d}\epsilon_{12}^e = \frac{\mathrm{d}\sigma_{12}}{2G},\ \mathrm{d}\epsilon_{11}^e = \frac{\mathrm{d}\sigma_{11}}{E},\ \mathrm{d}\epsilon_{12}^p = \hat{\sigma}_{12}\mathrm{d}\lambda = \sigma_{12}\mathrm{d}\lambda,\ \mathrm{d}\epsilon_{11}^p = \hat{\sigma}_{11}\mathrm{d}\lambda = \frac{2}{3}\sigma_{11}\mathrm{d}\lambda \tag{75}$$

Combining these findings:

$$\mathrm{d}\epsilon_{12} = \frac{\mathrm{d}\sigma_{12}}{G} + \sigma_{12}\mathrm{d}\lambda \quad \text{and} \quad \mathrm{d}\epsilon_{11} = \frac{\mathrm{d}\sigma_{11}}{E} + \frac{2}{3}\sigma_{11}\mathrm{d}\lambda. \tag{76}$$

Notice $\mathrm{d}\epsilon_{12} = 0$. Use the Mieses yield condition to eliminate σ_{12}.

$$(\sigma_{11})^2 + 3(\sigma_{12})^2 = (\sigma_0)^2 \quad \text{and} \quad 2\sigma_{11}\mathrm{d}\sigma_{11} + 6\sigma_{12}\mathrm{d}\sigma_{12} = 0 \tag{77}$$

$$\text{thus} \quad \mathrm{d}\epsilon_{11} = \frac{\left(\frac{E}{3} - G\right)\sigma_{11}^2 + G\sigma_0^2}{EG\left(\sigma_0^2 - \sigma_{11}^2\right)} \tag{78}$$

Integrate $\mathrm{d}\epsilon_{11} \in (0, \epsilon_{11})$.

8.3 Torsion of Prismatic Sections

8.3.1 St. Venant Torsion

Assumptions: Sections rotate as rigid bodies. If sections do warp they do so by the same amount.

Formulation
Kinematic Description: Small Sectional Rotation. Given θ the angle of twist per unit length. The angle of twist at any section is $\alpha = \theta x_3$. Where $\vec{e}_3$ is the along the neutral axis of the twisting rod. If α is small at every section perpendicular to the neutral axis, described by $\{\vec{e}_1, \vec{e}_2\}$ then:

$$u_1 = -\alpha x_2 = -\theta x_2 x_3 \quad \text{and} \quad u_2 = \alpha x_1 = \theta x_1 x_3. \tag{79}$$

The warping is independent of x_3 due to the assumptions:

$$u_3 = \theta\psi(x_1, x_2). \tag{80}$$

Small Strain Formulation

$$\widetilde{\epsilon} = \frac{1}{2}\left(\nabla\vec{u} + \vec{u}\nabla\right) \tag{81}$$

Substituting $\vec{u} = \{-\theta x_2 x_3, \theta x_1 x_3, \theta\psi(x_1, x_2)\}|_{\vec{e}}$ into the equation.

$$\begin{aligned} \epsilon_{11} = \frac{\partial u_1}{\partial x_1} &= 0 \\ \epsilon_{22} = \frac{\partial u_2}{\partial x_2} &= 0 \\ \epsilon_{33} = \frac{\partial u_3}{\partial x_3} &= 0 \end{aligned} \tag{82}$$

$$\begin{aligned} \epsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) &= \frac{1}{2}\theta\left(x_3 - x_3\right) = 0 \\ \epsilon_{23} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) &= \frac{1}{2}\theta\left(x_1 - \frac{\partial\psi}{\partial x_2}\right) \\ \epsilon_{13} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) &= \frac{1}{2}\theta\left(\frac{\partial\psi}{\partial x_1} - x_2\right) \end{aligned} \tag{83}$$

Constitutive Relation: Assume linear elastic isotropic material

$$\sigma_{23} = \mu 2\epsilon_{23} = \mu\theta\left(x_1 - \frac{\partial\psi}{\partial x_2}\right) \quad \text{and} \quad \sigma_{13} = \mu 2\epsilon_{13} = \mu\theta\left(\frac{\partial\psi}{\partial x_1} - x_2\right). \tag{84}$$

All other stress components (like the other strain components) are zero.

Equilibrium:

$$\nabla\cdot\widetilde{\sigma} = 0 \quad \text{static, body forces zero} \tag{85}$$

All equilibrium equations are identically zero except:

$$\frac{\partial}{\partial x_1}\left(\mu\theta\left(\frac{\partial\psi}{\partial x_1} - x_2\right)\right) + \frac{\partial}{\partial x_2}\left(\mu\theta\left(x_1 + \frac{\partial\psi}{\partial x_2}\right)\right) = 0. \tag{86}$$

Unless the solution is trivial $\mu = 0$ or $\theta = 0$ then $\nabla^2\psi = 0$.

Boundary Conditions: Traction on the outside of the member is zero. Forces on the front and back are equal and opposite. The normal vector can be separated into cartesian components

$$\vec{n} = n_1\vec{e}_1 + n_2\vec{e}_2. \tag{87}$$

Traction on the cross-section is:

$$\vec{t}_n = \widetilde{\sigma}\vec{n} = \widetilde{\sigma}\vec{e}_3 = \sigma_{j3}\vec{e}_j = \sigma_{13}\vec{e}_1 + \sigma_{23}\vec{e}_2 + \sigma_{33}\vec{e}_3 \tag{88}$$

Thus: $\vec{t}_n = \mu\theta\left(\frac{\partial\psi}{\partial x_1} - x_2\right)\vec{e}_1 + \mu\theta\left(x_1 + \frac{\partial\psi}{\partial x_2}\right)\vec{e}_2 = 0$. Expect no resultant load in the $\vec{e}_1$ and $\vec{e}_2$ directions.

$$\int_S \vec{t}_n \cdot \vec{e}_1 \mathrm{d}s = 0 \quad \text{and} \quad \int_S \vec{t}_n \cdot \vec{e}_2 \mathrm{d}s = 0 \tag{89}$$

Thus:

$$\int_A \left(x_1 + \frac{\partial\psi}{\partial x_2}\right)\mathrm{d}A \quad \text{and} \quad \int_A \left(\frac{\partial\psi}{\partial x_1} - x_2\right)\mathrm{d}A. \tag{90}$$

Solving $\int_A \left(x_1 + \frac{\partial\psi}{\partial x_2}\right)\mathrm{d}A$:

$$\begin{aligned}
&\int_A \left(\frac{\partial}{\partial x_2}\left(x_1x_2 + x_2\frac{\partial\psi}{\partial x_2}\right) - x_2\frac{\partial^2\psi}{\partial x_2^2}\right)\mathrm{d}A = \\
&\quad\int_A \left(\frac{\partial}{\partial x_2}\left(x_1x_2 + x_2\frac{\partial\psi}{\partial x_2}\right) + \frac{\partial}{\partial x_1}\left(x_2\frac{\partial\psi}{\partial x_1} + x_2^2\right)\right)\mathrm{d}A = \\
&\qquad\qquad\oint_c x_2\left(x_1n_2 + \frac{\partial\psi}{\partial x_2}n_2 + n_1\frac{\partial\psi}{\partial x_1} + n_1x_2\right)\mathrm{d}S = 0.
\end{aligned} \tag{91}$$

Similarly for the other reaction which is also zero.

The moment $M = \int_A (\sigma_{23}x_1 - \sigma_{13}x_2)\,\mathrm{d}A$.

$$M = \int_A \mu\theta\left[\left(x_1^2 + \frac{\partial\psi}{\partial x_2}x_1\right) + \left(x_2^2 - x_2\frac{\partial\psi}{\partial x_2}\right)\right]\mathrm{d}A \tag{92}$$

$$M = \mu\theta\mathbf{J} + \mu\theta\int_A \left(x_1\frac{\partial\psi}{\partial x_2} - x_2\frac{\partial\psi}{\partial x_1}\right)\mathrm{d}A. \tag{93}$$

Stress Potential Φ

Differentiate the stress equations given in equation [84] to derive the following.

$$\nabla^2\Phi = -2G\theta \tag{94}$$

8.4 Application to Cross Sections

The only cross section which satisfies a no warping assumption is a circular cross-section.

$$x_1^2 + x_2^2 = \text{constant} \quad \text{and} \quad \psi = 0 \tag{95}$$

8.4.1 Rectangular Shaft

Consider a rectangle bounded by $x = \pm\frac{a}{2}$ and $y = \pm\frac{b}{2}$ with $b > a$. Assume that:

$$\phi_1 = G\left[\frac{a^2}{2} - x^2\right] \tag{96}$$

This assumption satisfies the boundary condition only on the longer side. Assume a correction in fourier series form.

$$\phi_2 = G\theta \sum_{m=0}^{\infty} f_m(y) \cos\frac{(2m+1)\pi x}{a} \tag{97}$$

The correction should vanish in the stress potential equation:

$$\nabla^2\Phi_2 = G\theta \sum_{m=0}^{\infty} \left[f_m''(y) - \frac{(2m+1)^2\pi^2}{a^2} f_m(y)\right] \cos\frac{(2m+1)\pi x}{a}. \tag{98}$$

The easiest way to achieve this is:

$$f_m''(y) = \frac{(2m+1)^2\pi^2}{a^2} f_m(y) \quad \text{and} \quad f_m(y) = A_m \cosh\frac{(2m+1)\pi y}{a}, \tag{99}$$

The boundary condition still needs to be satisfied. In this coordinate system $\pi_2 = -\phi_1$ on $y = \pm\frac{b}{2}$,

$$\frac{a^2}{4} - x^2 = \sum_{m=0}^{\infty} B_m \cos\frac{(2m+1)\pi x}{a} \tag{100}$$

Using orthogonality of cosines:

$$B_m = \frac{4}{a}\int_0^{\frac{a}{2}} \left[\frac{a^2}{4} - x^2\right] \cos\frac{(2m+1)\pi x}{a} \mathrm{d}x = \frac{8(-1)^m a^2}{(2m+1)^3\pi^3}. \tag{101}$$

To satisfy the boundary condition $A_m \cosh[2(m+1)\pi b/2a] = -Bm$. Thus:

$$\phi_2 = -G\theta\frac{8a^2}{\pi^3} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^3} \frac{\cosh\frac{(2m+1)\pi y}{a}}{\cosh\frac{(2m+1)\pi b}{2a}} cos\frac{(2m+1)\pi x}{a}. \tag{102}$$

The integration leads to

$$J = a^3 b\left[\frac{1}{3} - \frac{64a}{\pi^5 b}\sum_{m=0}^{\infty} \frac{1}{(2m+1)^5} \tanh\frac{(2m+1)\pi b}{2a}\right]. \tag{103}$$

The series converges very rapidly.

$$\tau_{\max} = Ga\theta\left[1 - \frac{8}{\pi^2}\sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} \mathrm{sech}\frac{(2m+1)\pi b}{2a}\right]. \tag{104}$$

For very thin sections $\frac{b}{a} \to \infty$:

$$J \approx \frac{1}{3}\sum_{i=1}^{n} a_i^3 b_i \quad \text{and} \quad \tau_{\max} \approx \frac{Ta_{\max}}{J} \tag{105}$$

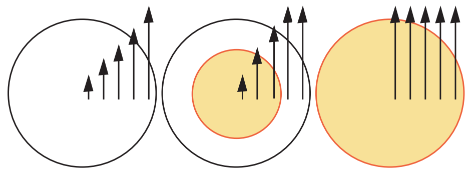


Figure 4: Elastic and Plastic Torsion Stress distribution

8.4.2 Plastic Torsion

Mathematically the stress potential function for a bar in fully plastic torsion is

$$|\nabla\Phi| = k \quad \text{in} \quad A, \qquad \text{and} \qquad \phi = 0 \quad \text{on} \quad C \tag{106}$$

It has a unique solution $\Phi_p(x_1, x_2)$. There is a discontinuity even at $\Theta \to \infty$ of the plastic solution. In a circular cross-section the discontinuity is at the center, where the direction of stress acting at the point is undefined. For a rectangular polygon the points with undefined directions include the corners, a line in the center – the plastic surface is the form of a pyramid. The ultimate torque then can be given by

$$T_U = 2\int_A \phi_p \mathrm{d}A \tag{107}$$

For a circle of radius a $T_U = \frac{2}{3}\pi k a^3$. for a rectangle of sides a, b, with $b > a$: $T_U = \frac{1}{6}ka^2(3b - a)$.

8.4.3 Contained Plastic Torsion

For values of the twist θ such that $\theta_E \leq \theta \leq \infty$, the cross-sectional area A consists of elastic and plastic regions governing the stress function θ see figure 4.

To determine the behavior of the elastic region its boundary must be specified explicitly, thus the plastic portion must be solved first to discover this boundary. Also the elastic region satisfies the plastic conditions $\theta < \theta_p$ and $|\nabla\theta| < k$. The solution can be found by minimizing the functional

$$K[\phi^*] \hat{=} \int_A (|\nabla\phi^*|^2 - 4G\theta\phi^*)\mathrm{d}A \tag{108}$$

where the minimum is taken either over the functions ϕ^* that satisfy $\phi^* < \phi_p$ or $|\nabla\phi^*| \leq k$.

8.4.4 Circular Shaft

From symmetry the boundary must be at a constant radius, say r^*. Thus, in the elastic region $r < r^*$ in the plastic region $r > r^*$. c is the shaft radius. The fully plastic solution is

$$\phi = k(c - r). \tag{109}$$

The elastic is:

$$\nabla^2\phi = -2G\theta \quad \text{integrating} \quad -\phi_{,r} = \tau_{z\theta} = \tau = G\theta r \tag{110}$$

When $\tau = k$, $r^* = \frac{k}{G\theta}$ and $\theta_E = \frac{k}{Gc}$. Integrate again and find for $r < r*$

$$\phi(r) = k\left[c - \frac{r^*}{2} - \frac{r^2}{2r^*}\right] \tag{111}$$

and the torque is

$$T = \frac{2\pi}{3}k(c^3 - \frac{r^{*3}}{4}) = T_U\left[1 - \frac{1}{4}\left(\frac{\theta_E}{\theta}\right)^2\right] \tag{112}$$

8.5 Impact of a Rigid-Plastic Bar

The longitudinal impact of a rigid-plastic bar of length L, can be treated as a one-dimensional plastic wave problem. If the bar is moving rigidly at a speed of v_0 until it impacts with a rigid target at time $t = 0$. If the shock front is an infinitely narrow zone at $x = \xi(t)$. At impact the free end of the bar is moving at a velocity $\bar{v}(t)$ and the end of the bar which has impacted has come to rest:

$$v(x,t) = \bar{v}(t),\ x < \xi(t) \quad v(x,t) = 0,\ x > \xi(t) \tag{113}$$

Thus the shock speed c in the positive x direction is $c = -\dot{\xi}$ with respect to the undeformed material.

If the bar is rigid-perfectly plastic such that there is yield at the shock front. At the shock front $x = \xi(t)$ the strain σ is equal to the yield stress behind the front the bar has been deformed by $\bar{\sigma}$ and the new strain is $\frac{\sigma_y}{1-\bar{\epsilon}}$. The discontinuities at the front can thus be given as:

$$[\sigma] = \frac{\sigma_y}{(1-\bar{\epsilon})} - \sigma_y = \frac{\sigma_y\bar{\epsilon}}{1-\bar{\epsilon}}, \quad [\epsilon] = \bar{\epsilon}, \quad [v] = -\bar{v} \tag{114}$$

Thus, the shock relations are:

$$\bar{\epsilon} = -\frac{\bar{v}}{\dot{\xi}},\ \frac{\sigma_y\bar{\epsilon}}{1-\bar{\epsilon}} = -\rho\dot{\xi}\bar{v} \quad \text{and} \quad \rho\dot{\xi}^2 = \frac{\sigma_y}{1-\bar{\epsilon}},\ \rho\bar{v}^2 = \frac{\sigma_y\bar{\epsilon}^2}{1-\bar{\epsilon}} \tag{115}$$

The initial strain in terms of the impact velocity is

$$\frac{\rho v_0^2}{\sigma_y} = \frac{\epsilon_0^2}{1-\epsilon_0}. \tag{116}$$

Differentiate and use the equation of motion of the undeformed portion to find:

$$\rho\bar{v}\mathrm{d}\bar{v} = \frac{\sigma_y}{2}\mathrm{d}\left(\frac{\bar{\epsilon}^2}{1-\bar{\epsilon}}\right) = -\frac{\sigma_y\bar{v}}{\xi} = \sigma_y\bar{\epsilon}\frac{\dot{\xi}}{\xi} \tag{117}$$

Rearrange and integrate to find the jump in strain.

$$\ln\left(\frac{\xi}{L}\right)^2 = \frac{1}{1-\bar{\epsilon}} - \frac{1}{1-\epsilon_0} - \ln\frac{1-\bar{\epsilon}}{1-\epsilon_0}. \tag{118}$$

The position of the shock front after impact occurs when the jump in strain is zero.

$$\zeta_0 = L\sqrt{1-\epsilon_0}\ \mathrm{e}^{-\frac{\epsilon_0}{2(1-\epsilon_0)}} \tag{119}$$

Thus the portion $0 < x < \xi_0$ remains undeformed. The same result was found by Taylor in 1948 based on a different approach using an approximated formula for dynamic yield stress. A similar approach can be applied to work hardened material.

8.6 Other Exercises

8.6.1 Strain Example

$$z_1 = x_1\left(1 + \frac{x_2 * x + 3}{\alpha}\right),\ z_2 = x_2,\ z_3 = x_3 \tag{120}$$

$$\nabla\vec{z};\ \frac{\partial z_i}{\partial x_a} = \begin{bmatrix} 1 + \frac{x_2 * x_3}{\alpha} & \frac{x_1 * x_3}{\alpha} & \frac{x_1 * x_2}{\alpha} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{121}$$

Recall the large strain tensor in cartesian coordinates:

$$\epsilon_{ab}(x,t) = \frac{1}{2}\left[z_{k,a}z_{k,b} - \delta_{ab}\right] = \frac{1}{2}\left[(z_{1,a}z_{1,b}) + (z_{2,a}z_{2,b}) + (z_{3,a}z_{3,b}) - \delta_{ab}\right] \tag{122}$$

Solving for the strain components:

$$\begin{aligned}
\epsilon_{11} &= \frac{1}{2}\left[\left(1 + \frac{x_2 * x_3}{\alpha}\right)^2 - 1\right] \\
\epsilon_{22} &= \frac{1}{2}\left[\left(\frac{x_1 * x_3}{\alpha}\right)^2\right] \\
\epsilon_{33} &= \frac{1}{2}\left[\left(\frac{x_1 * x_2}{\alpha}\right)^2\right] \\
\epsilon_{12} &= \frac{1}{2}\left[\left(1 + \frac{x_2 * x_3}{\alpha}\right)\left(\frac{x_1 * x_3}{\alpha}\right)\right] \\
\epsilon_{13} &= \frac{1}{2}\left[\ \left(1 + \frac{x_2 * x_3}{\alpha}\right)\left(\frac{x_1 * x_2}{\alpha}\right)\right] \\
\epsilon_{23} &= \frac{1}{2}\left[\ \left(\frac{x_1 * x_3}{\alpha}\right)\left(\frac{x_1 * x_2}{\alpha}\right)\right]
\end{aligned} \tag{123}$$

Appendix: Elasticity

A.1 References

- Foundations of Solid Mechanics [4]

- Introduction to Continuum Mechanics [6]
- Elasticity, Prof. Gerard Atesian, Columbia University, E???, Fall 1999
- Elasticity, Prof. Rene B. Testa, Columbia University, E??, Spring 2000

A.2 Glossary

Gibbs Notation: A vector notation which is invariant of coordinate systems.

Summation Convention: the repetition of an index (whether superscript or subscript) in a term denotes a summation with respect to that index over its range.

The summation convention can also be used for differentiation.

$$df = \frac{\partial f}{\partial x^i} dx^i \tag{124}$$

Range: The range of an index i is the set of n integer values 1 to n.

Dummy Index: an index which is summed over.

Free Index: an index which is not summed over.

Kronecker Delta: $\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

Permutation Symbol: $e_{ijk} = \begin{cases} 1 & even \\ -1 & odd \\ 0 & neither \end{cases}$

The permutation symbol and the Kronecker Delta are related as follows.

$$e_{ijk}e_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} \tag{125}$$

Cartesian Tensors: Covariance and contravariance are equal. The transformation tensor for cartesian vectors is an orthogonal tensor Q. Orthogonality preserves the lengths and angles of the transformed vectors. Reflection, rotation, and translation are orthogonal translations. Also $Q^T Q = I$

Contraction: The process of equating and summing a covariant and contravariant index of a mixed tensor. the result of contraction is another tensor, if no free index is left the resulting quantity is a scalar.

$$\frac{\partial a^\beta}{\partial a^\alpha} = \delta^\beta_\alpha \tag{126}$$

Quotient Rule: The Quotient Rule can be used to determine if a function is a tensor without determining the law of transformation directly.

Theorem 1: If $[A(i_1, i_2, i_3, \ldots, i_r)]$ is a set of functions of the variables x^i, and if the product $A(\alpha, i_2, i_3, \ldots, i_r)\xi^\alpha$ with an arbitrary vector ξ^α be a tensor of the type $A^{j_1 \ldots j_p}_{k_1 \ldots k_q}(x)$, then the set $A(i_1, i_2, i_3, \ldots, i_r)$ represents a vector of the type $A^{j_1 \ldots j_p}_{\alpha k_1 \ldots k_q}(x)$.

Theorem 2: Similarly, if the product of a set of n^2 functions $A(\alpha, j)$ with an arbitrary tensor $B_{\alpha k}$ (and summed over α) is a covariant tensor of rank 2, then $A(i, j)$ represents a tensor of the type A^i_j.

Physical Components of a Vector: The base vectors g_r and g^r are in general not unit vectors. Their lengths are:

$$|\mathbf{g}_r| = \sqrt{g_{rr}}, \quad |\mathbf{g}^r| = \sqrt{g^{rr}}, \quad \text{r not summed} \tag{127}$$

$$\mathbf{v} = \sum_{r=1}^{3} v^r \sqrt{g_{rr}} \frac{\mathbf{g}_r}{\sqrt{g_{rr}}} = \sum_{r=1}^{3} v_r \sqrt{g^{rr}} \frac{\mathbf{g}^r}{\sqrt{g^{rr}}} \tag{128}$$

Then, since $\mathbf{g}_r/\sqrt{g_{rr}}$ and $\mathbf{g}^r/\sqrt{g^{rr}}$ are unit vectors, all components $v_r\sqrt{g^{rr}}$ and $v^r\sqrt{g_{rr}}$ (r not summed) will have the same physical dimensions. The physical components of a vector include the square root of a metric.

Euclidean Metric Tensor: $g_{km}(\theta_1, \theta_2, \theta_3)$ is a Measure of length in a reference system.

$$g_{km}(\theta_1, \theta_2, \theta_3) = \sum_{i=1}^{3} \frac{\partial x_i}{\partial \theta_k} \frac{\partial x_i}{\partial \theta_m}. \tag{129}$$

Given a line element dx^1, dx^2, dx^3. The length of the element ds is determined by Pythagoras' rule: $ds^2 = dx^i dx^i = \delta_{ij} dx^i dx^j$ and is independent of the coordinate system. If dx is transformed to another coordinate system where x_i is a function of new basis vectors $(\theta_1, \theta_2, \theta_3)$. The change in length dx is related to the transformed coordinate system as follows: $dx_i = \frac{\partial x_i}{\partial \theta_k} d\theta^k$.

Substitute this into *Pythagoras' Theorem.*

$$ds^2 = \sum_{i=1}^{3} \frac{\partial x_i}{\partial \theta_k} \frac{\partial x_i}{\partial \theta_m} d\theta^k d\theta^m = g_{km} d\theta^k d\theta^m \tag{130}$$

Notice, the *Euclidean Metric* is symmetric $g_{km} = g_{mk}$.

Principal Values: The principal values of a tensor lie on the principal planes. They are the maximum and minimum values of the system. Let $\mathbf{v}$ define a principal axis and let σ be the corresponding principal value. Then the vector acting on the surface normal to $\mathbf{v}$ has components $\tau_{ij} v_j$ and the components of only the principal components σv_i.

$$(\tau_{ij} - \sigma \delta_{ji}) v_j = 0 \tag{131}$$

Since τ_{ij} as a matrix is real and symmetric, therefore there exist three real-valued principal stresses and a set of orthonormal principal axes.

A solution to $\mathbf{v}$ has a nonzero solution if and only if

$$|\tau_{ij} - \sigma\delta_{ij}| = -\sigma^3 + \mathbf{I}_1\sigma^2 - \mathbf{I}_2\sigma + \mathbf{I}_3 = 0 \tag{132}$$

Tensor Invariants: $\mathbf{I}$ are the invariants of a tensor.

$$\begin{aligned}
\mathbf{I}_1 &= \tau_{11} + \tau_{22} + \tau_{33} \\
\mathbf{I}_2 &= \begin{vmatrix} \tau_{22} & \tau_{23} \\ \tau_{32} & \tau_{33} \end{vmatrix} + \begin{vmatrix} \tau_{11} & \tau_{13} \\ \tau_{31} & \tau_{33} \end{vmatrix} + \begin{vmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{11} \end{vmatrix} \\
\mathbf{I}_1 &= \begin{vmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{vmatrix}
\end{aligned} \tag{133}$$

The invariants can also be written in terms of principal values.

Strain Energy: The strain energy W is a scalar function of all the variables which cause strain in a material. For a linear elastic material it is a linear function of the strains: $\frac{\partial W}{\partial \epsilon_{ij}} = \sigma_{ij}$.

A.3 Tensors

A.3.1 Scalars, Contravariant Vectors, Covariant Vectors

In nonrelativistic physics there are quantities like mass and length which are independent of reference coordinates, these quantities are tensors.

Scalars, covariant vector fields, and contravariant vector fields are all examples of tensors.

A.3.2 Transformations of Tensors

Contravariant Tensor Field of Rank Two, t_{ij}:

$$\bar{t}_{ij}(\bar{\theta}^1, \bar{\theta}^2, \bar{\theta}^3) = t_{mn}(\theta^1, \theta^2, \theta^3)\frac{\partial \theta^m}{\partial \bar{\theta}^i}\frac{\partial \theta^n}{\partial \bar{\theta}^j} \tag{134}$$

Contravariant Tensor Field of Rank two, t^{ij}:

$$\bar{t}^{ij}(\bar{\theta}^1, \bar{\theta}^2, \bar{\theta}^3) = t^{mn}(\theta^1, \theta^2, \theta^3)\frac{\partial \bar{\theta}^i}{\partial \theta^m}\frac{\partial \bar{\theta}^j}{\partial \theta^n} \tag{135}$$

Mixed tensor field of Rank Two, t^i_j:

$$\bar{t}^i_j(\bar{\theta}^1, \bar{\theta}^2, \bar{\theta}^3) = t^m_n(\theta^1, \theta^2, \theta^3)\frac{\partial \bar{\theta}^i}{\partial \theta^m}\frac{\partial \theta^n}{\partial \bar{\theta}^j} \tag{136}$$

Transformations of the kronecker delta and permutation symbol tensors in general coordinates:

$$g_{ij} = \frac{\partial x^m}{\partial \theta^i}\frac{\partial x^n}{\partial \theta^j}\delta_{mn} = \frac{\partial x^n}{\partial \theta^i}\frac{\partial x^n}{\partial \theta^j} \tag{137}$$

$$g^{ij} = \frac{\partial x^i}{\partial \theta^m}\frac{\partial x^j}{\partial \theta^n}\delta^{mn} = \frac{\partial x^i}{\partial \theta^n}\frac{\partial x^j}{\partial \theta^n} \tag{138}$$

$$g^i_j = \frac{\partial x^i}{\partial \theta^m}\frac{\partial x^n}{\partial \theta^j}\delta^m_n = \delta^i_j \tag{139}$$

$$\epsilon_{ijk} = \frac{\partial x^r}{\partial \theta^i}\frac{\partial x^s}{\partial \theta^j}\frac{\partial x^t}{\partial \theta^k}e_{rst} = e_{ijk}\left|\frac{\partial x^m}{\partial \theta^n}\right| = e_{ijk}\sqrt{g} \tag{140}$$

$$\epsilon^{ijk} = \frac{\partial \theta^i}{\partial x^r}\frac{\partial \theta^j}{\partial x^s}\frac{\partial \theta^k}{\partial x^t}e^{rst} = e^{ijk}\left|\frac{\partial \theta^m}{\partial x^n}\right| = \frac{e^{ijk}}{\sqrt{g}} \tag{141}$$

where g is the value of the determinant $|g_{ij}|$ and is positive for any proper coordinate system $g = |g_{ij}| > 0$.

Tensor fields of higher ranks: A field with contravariant rank p and covariant rank q and the rank of the tensor is $r = p + q$. The components in any two coordinate systems is related by.

$$\bar{t}^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q} = \frac{\partial\bar{\theta}^{\alpha_1}}{\partial\theta^{k_1}}\cdots\frac{\partial\bar{\theta}^{\alpha_p}}{\partial\theta^{k_p}}\cdot\frac{\partial\theta^{m_1}}{\partial\bar{\theta}^{\beta_1}}\cdots\frac{\partial\theta^{m_q}}{\partial\bar{\theta}^{\beta_q}}t^{k_1\ldots k_p}_{m_1\ldots m_q} \tag{142}$$

Tensor Theorem: Let $A^{\beta_1\ldots\beta_s}_{\alpha_1\ldots\alpha_r}$, $B^{\beta_1\ldots\beta_s}_{\alpha_1\ldots\alpha_r}$ be tensors. The equation

$$A^{\beta_1\ldots\beta_s}_{\alpha_1\ldots\alpha_r}(\theta^1,\theta^2,\ldots,\theta^n) = B^{\beta_1\ldots\beta_s}_{\alpha_1\ldots\alpha_r}(\theta^1,\theta^2,\ldots,\theta^n) \tag{143}$$

is a tensor equation. Thus if it is true in some coordinate system then it is true in all coordinate systems which are in one-to-one correspondence with each other.

A.3.3 Tensor Products

The product of tensors is a tensor since it transforms like a tensor.

Dot Product: The dot product reduces the order of the tensors involved: $\vec{a}\cdot\vec{b} = a^i\vec{g}_i\cdot b_j\vec{g}^j = a^ib_j(\vec{g}_i\cdot\vec{g}^j) = a^ib_i$.

Cross Product: The cross product preserves the order of the tensors involved: $\vec{a}\times\vec{b} = a^ib_j(g_i\times b^j)$. For cartesian coordinates: $\vec{a}\times\vec{b} = \epsilon_{ijk}a_ib_j\vec{e}_k$. See the transformation of permutation symbol in the glossary.

Dyadic: The dyadic increases the order of the tensors involved. $a_ib_j = T_{ij}$

A.3.4 Covariant Differentiation of Vector Fields

Differentiation in generalized coordinates caries an extra term. Without this extra term the derivative is not a tensor.

$$\xi^i|_j = \frac{\partial \xi^i}{\partial x^j} + \Gamma(i,j,\alpha)\xi^\alpha \tag{144}$$

Euclidean Christoffel Symbols:

$$\Gamma^i_{\alpha\beta}(x^1,x^2,x^3) = \frac{1}{2}g^{i\sigma}(\frac{\partial g_{\sigma\beta}}{\partial x^\alpha} + \frac{\partial g_{\alpha\sigma}}{\partial x^\beta} - \frac{\partial g_{\alpha\beta}}{\partial x^\sigma}) \tag{145}$$

The Euclidean Christoffel Symbol does not transform as a tensor. It transforms as follows:

$$\bar{\Gamma}^i_{\alpha\beta}(\bar{x}^1,\bar{x}^2,\bar{x}^3) = \bar{\Gamma}^\lambda_{\mu\nu}(x^1,x^2,x^3)\frac{\partial x^\mu}{\partial \bar{x}^\alpha}\frac{\partial x^\nu}{\partial \bar{x}^\beta}\frac{\partial \bar{x}^i}{\partial x^\lambda} + \frac{\partial^2 x^\lambda}{\partial \bar{x}^\alpha \partial \bar{x}^\beta}\frac{\partial \bar{x}^i}{\partial x^\lambda} \tag{146}$$

Thus,the christoffel term is required for the derivative in general coordinates to transform as a tensor:

$$\frac{\partial \bar{\xi}^i}{\partial \bar{x}^a} + \bar{\Gamma}^i_{m\alpha}\bar{\xi}^m = \left(\frac{\partial \xi^\lambda}{\partial x^\mu} + \Gamma^\lambda_{s\mu}\xi^s\right)\frac{\partial x^\mu}{\partial \bar{x}^\alpha}\frac{\partial \bar{x}^i}{\partial x^\lambda} \tag{147}$$

Thus, $\xi^i|_\alpha$ is a covariant derivative of a contravariant tensor. The covariant derivative of a covariant vector field is written as follows.

$$\xi_i|_a = \frac{\partial \xi_i}{\partial x^\alpha} - \Gamma^\sigma_{i\alpha}\xi_\sigma \tag{148}$$

More generally the covariant derivative of a tensor $T^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q}$ of rank $p+q$, contravariant of rank p, covariant of rank q can be written as follows

$$\begin{aligned} T^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q}|_\gamma &= \frac{\partial T^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q}}{\partial x^\gamma} + \Gamma^{\alpha_1}_{\sigma_\gamma}T^{\sigma\alpha_2\ldots\alpha_p}_{\beta_1\beta_2\ldots\beta_q} + \ldots \\ &\Gamma^{\alpha_p}_{\sigma_\gamma}T^{\alpha_1\ldots\alpha_{p-1}\sigma}_{\beta_1\ldots\beta_{q-1}\beta_q} - \Gamma^\sigma_{\beta_1\gamma}T^{\alpha_1\alpha_2\ldots\alpha_p}_{\sigma\beta_2\ldots\beta_q} - \ldots \\ &-\Gamma^\sigma_{\beta_q\gamma}T^{\alpha_1\ldots\alpha_{p-1}\alpha_p}_{\beta_1\ldots\beta_{q-1}\sigma} \end{aligned} \tag{149}$$

This derivative is contravariant of rank p, and covariant of rank $q+1$. Notice that the christoffel symbols are zero in the cartesian coordinates.

A.4 Kinematics

Lagrangian: description of motion is constructed by following individual particles *particle point of view* — Spatial description.

$$\vec{r} = \vec{f}^l(\vec{x}^0, t) \tag{150}$$

Eulerian: description of motion constructed by observing the passage of particles through a fixed position in space the *field point of view* — Material description.

$$\vec{r} = \vec{f}^e(\vec{x}, t) \tag{151}$$

There is a correspondence between the Lagrangian and Eulerian description. The particle which arrives in the Eulerian observed field at time t can be described by its initial condition $\vec{x}^0$ (or marker or color) using the Lagrangian description: $x_i = f_i^l(\vec{x}^0, t)$. The field position is now described in terms of the markers of the particle and the Eulerian description can be formulated in the Lagrangian perspective: $x_i = f_i^e(f_1^l(\vec{x}, t), f_2^l(\vec{x}, t), f_3^l(\vec{x}, t), t)$. The Lagrangian can be solved using an inverse procedure.

A.5 Kinetics

The Governing Equations of Continuum Mechanics describe the behavior of a body which remains continuous under the action of external forces ([4]), and satisfies the following assumptions.

Meso Scale: The material which makes up the body can be described by its average properties on a scale larger than the micro scale but smaller than the macro scale ([8]). For example, average energy remains constant independent of average or other mean value.

Force at a Point: The limit with respect to the meso scale:

$$\lim_{\delta a \to 0} \frac{\delta P}{\delta A} = 0, \quad \text{where } \delta\text{P is the net force that acts on the area } \delta\text{A} \tag{152}$$

Moment at a Point: In materials with high stress gradients the moment at a point cannot be considered. However, for most material the following limit in the meso scale is acceptable:

$$\lim_{\delta a \to 0} \frac{\delta M}{\delta A} = 0, \quad \text{where } \delta\text{M is the net moment that acts on the area } \delta\text{A} \tag{153}$$

Materials which do not have this behavior are considered in theorems related to the *Cosserat Medium*.

Points on the body can be described with respect to material or spatial reference systems. Material equations of state describe the body in terms of its initial configuration (*Lagrangian:* z^k). Spatial equations describe the body in terms of its final configuration (*Eulerian:* x^a).

In material form the conservation of mass is $\rho_0 = \rho\mathbf{J}$ so $\rho = \rho_0 \frac{|g_{ij}|^{1/2}}{|g_{bc}|^{1/2}} \left|\frac{\partial z^k}{\partial x^a}\right|$.

Conservation of Mass: Mass can neither be created or destroyed. Thus, the change of mass in a control volume (cv) is equal to the amount of mass convected into it through its surface (s) with normal $\vec{n}$.

$$\frac{\partial}{\partial t}\int_{cv}\rho \ \mathrm{dV} = -\int_{\mathrm{s}}\rho\vec{\mathrm{v}}\cdot\vec{\mathrm{n}} \ \mathrm{dS} \tag{154}$$

Using *Green's Theorem* and considering that by definition the control volume does not change with time.

$$\frac{\partial\rho}{\partial t} = \ -\nabla\cdot\rho\vec{v} = \ -\vec{v}\cdot\nabla\rho \ + \ -\rho\nabla\cdot\vec{v} \tag{155}$$

$$\frac{\mathrm{D}\rho}{\mathrm{Dt}} + \rho(\nabla\cdot\vec{v}) = 0. \tag{156}$$

Conservation of Linear Momentum: Surface forces and body forces are the only sources of linear momentum. The change of linear momentum in a control volume (cv) is equal to the linear momentum convected into the volume over its surface (s) described by the normal $\vec{n}$, the traction tensor ($\widetilde{P}$) and the body forces ($\vec{b}$).

$$\begin{aligned}\tfrac{\partial}{\partial t}\textstyle\int_{cv} \ \rho\vec{v} \ \mathrm{d}V = -\textstyle\int_s(\rho\vec{v}) \ \vec{v}\cdot\vec{n} \ \mathrm{d}S \\ +\textstyle\int_S\widetilde{P}\cdot\vec{n} \ \mathrm{d}S + \textstyle\int_{cv}\rho\vec{b} \ \mathrm{d}V\end{aligned} \tag{157}$$

Following the procedure used for conservation of mass,

$$\frac{\partial\rho\vec{v}}{\partial t} = -\nabla\cdot(\rho\vec{v}\vec{v}) + \nabla\cdot\widetilde{P} + \rho\vec{b} \tag{158}$$

Using conservation of mass:

$$\rho\frac{\mathrm{D}\vec{v}}{\mathrm{D}t} = \nabla\cdot\widetilde{P} + \rho\vec{b}. \tag{159}$$

Conservation of Angular Momentum: Following the same procedure as for linear momentum and using previous laws, this conservation law requires that the stress tensor ($\widetilde{P}$) be symmetric when it describes a non-*Cosserat Medium*.

Conservation of Energy: Energy can neither be created or destroyed only changed in form. The change of kinetic energy and internal energy (e) in a control volume (cv) is equal to the work done on the surface by the traction ($\widetilde{P}$) and by the body forces ($\vec{b}$) and the heat convected into the volume and heat created by body($\vec{r}$) ([6]).

$$\begin{aligned}\tfrac{\mathrm{D}}{\mathrm{D}t}\textstyle\int_{cv}(\rho\frac{v^2}{2} + \rho e) \ \mathrm{d}V = \textstyle\int_s(\widetilde{P}\cdot\vec{n})\cdot\vec{v} \ \mathrm{d}S \\ +\textstyle\int_{cv}\rho\vec{b}\cdot\vec{v} \ \mathrm{d}V - \textstyle\int_s\vec{q}\cdot\vec{n} \ \mathrm{d}S + \textstyle\int_{vc}\rho\vec{r}\end{aligned} \tag{160}$$

Using previous equations and methods:

$$\rho\frac{\mathrm{De}}{\mathrm{D}t} = \mathrm{tr}(\widetilde{\mathrm{P}}\nabla\vec{\mathrm{v}}) - \nabla\cdot\vec{\mathrm{q}} + \rho\vec{\mathrm{r}}. \tag{161}$$

First Law of Thermodynamics: Using conservation of energy: the system change in kinetic energy K and potential energy U is equal to the work done and the increase in energy in the system.

$$\dot{\bar{K}} + \dot{\bar{U}} = \bar{P} + \bar{Q} \tag{162}$$

Second Law of Thermodynamics: Given in cartesian coordinates:

$$\rho T\dot{\bar{S}} + q_{i,i} - \frac{1}{T}q_{i,i}T_{,i} - \rho r \geq 0 \tag{163}$$

S is the entropy and T is the temperature of the system. A process is *reversible* if the right hand side is equal to zero.

Potentials An energy potential is a partial derivative of the total system potential energy with respect to a system variable. In general:

$$\tau_k = \frac{\partial U}{\partial \alpha_k} \tag{164}$$

Helmholz Free Energy: Another representation of the energy potential

$$A = U - TS \tag{165}$$

The second law of thermodynamics is thus defined as:

$$\sigma_{ij}\dot{\epsilon}_{ij} - \rho(\dot{A} + \dot{T}S) - \frac{1}{T}qT_{,i} \geq 0 \tag{166}$$

Energy of an Elastic Body: The potential energy in an elastic body depends only on the strain and the temperature $A(\epsilon_{ij}, T)$.
The first law of thermodynamics reduces to: $\sigma_{ij} - \rho\frac{\partial A}{\partial \epsilon_{ij}} = 0$.
The second law of thermodynamics reduces to: $\frac{1}{T}q_i T_{,i} \leq 0$.

Potential Functions: are used to simplify the description of stress, the potential of an elastic body. These functions satisfy Equilibrium and Compatibility conditions automatically. [1] Boundary conditions must still be considered.

A.6 Stress & Strain Definitions

A.6.1 Strain

Strain is defined as elongation per unit length. The *displacement vector*r: $\vec{u} = \vec{r} - \vec{r}o + \vec{d}$. $\vec{r}_0$ is the vector to the undeformed $\vec{r}$ is the vector to the deformed

[1] Maxwell Moreial: $\sigma = \nabla \times \Phi \times \nabla$; Airy Stress function: $\sigma_{11} = \phi_{,22}, \sigma_{22} = \phi_{,11}$ such that $\nabla^4\phi = 0$ is the only remaining governing equation; and the Torsion function: $\sigma = \nabla \cdot \psi$ see section on St. Venant Torsion. Plane stress and plane strain can also be represented by the airy stress function using the Beltrami-Michell Equation.

position. $\vec{d}$ is the distance between the origins of the undeformed and deformed configurations, the origins are fixed.

$$\mathrm{d}\vec{u} = \mathrm{d}\vec{r} - \mathrm{d}\vec{r}_0 \tag{167}$$

By definition: the finial configuration is defined in terms of *spatial* coordinates $\vec{r} = z^i\vec{g}_i$, while the original is defined in terms of *material* $\vec{r}_0 = x^a\vec{g}_a$. Thus the change in the displacement vector can be written in terms of its derivative in the *spatial* or *material* coordinate system.

$$u_{i|k}\mathrm{d}z^k = \mathrm{d}z^i - \mathrm{d}x^{a=i} \tag{168}$$

$$u_{a|b}\mathrm{d}x^b = \mathrm{d}z^{i=a} - \mathrm{d}x^a \tag{169}$$

The strain tensor is a defined quantity. It is described by the changes in length of the vectors $\vec{r}_0$ and $\vec{r}$. Set $\mathrm{d}l = |\mathrm{d}r|$ and $\mathrm{d}l_0 = |\mathrm{d}r_0|$.

$$\mathrm{d}l^2 - \mathrm{d}l_0^2 = 2e_{ab}\mathrm{d}x^a\mathrm{d}x^b = 2h_{kl}\mathrm{d}z^k\mathrm{d}z^l \tag{170}$$

Solving for the *Green Strain Tensor* and the *Cauchy Strain Tensor* respectively:

$$e_{ab}(x,t) = \frac{1}{2}\left[g_{kl}\ z^k|_a z^l|_b - g_{ab}\right] \qquad h_{kl}(z,t) = \frac{1}{2}\left[g_{kl} - g_{ab}\ x^a|_k x^b|_l\right]. \tag{171}$$

Notice that e_{ab} and h_{kl} are symmetric tensors. In terms of deformation:

$$e_{ab}(x,t) = \frac{1}{2}\left[u_{c|a}u^c|_b + u_{b|a} + u_{a|b}\right] \qquad h_{kl}(z,t) = \frac{1}{2}\left[u_{k|l} + u_{l|k} - u_{m|k}u^m|_l\right] \tag{172}$$

Assuming small displacement gradients $e_{ab} \approx h_{kl}$.

A.6.2 Stress

Stress is the force per unit area. *Traction* is the force at a point. The *stress tensor* when operated upon the unit normal to a surface reveals the traction on that surface: $\widetilde{\sigma} \cdot \vec{n} = \vec{t}_n$. Stress tensors are defined in terms of initial and final configurations and forces.

name	**symbol**	**physical characteristic**	**symmetric**
	$\widetilde{P}^{kl}$	final forces on final area	yes
Piolla Kirchhoff I	$\widetilde{S}^{ab}$	initial forces on initial area	no
Piolla Kirchhoff II	$\widetilde{t}^{ab}$	$S_{ab} = t_{al}x^b_{,l}$ & $t^{al} = S^{ab}z^l_{,b}$	yes

If the stress tensor satisfies static equilibrium in a non-cosserat medium, it is symmetric. Proof of the symmetry can be shown using the conservation of angular momentum.

A.6.3 Constitutive Equations

The relationship between stress and strain is empirically defined. Either it is assumed to be linear elastic, or the contributions to the strain energy of the system are assumed to be conservative and dependent only on first order of strain.

Elastic: Let the strain energy function be defined only in terms of strain and temperature $W(\epsilon_{ij}, t)$. Thus strain is defined as follows:

$$\sigma_{ij} = \frac{\mathrm{d}W}{\mathrm{d}\epsilon_{ij}} = \frac{\partial W}{\partial \epsilon_{ij}} + \frac{\partial W}{\partial T}\frac{\partial T}{\partial \epsilon_{ij}} \tag{173}$$

Generalized Hooke's Law: The simplest form of a linear relationship between stress and strain is:

$$\sigma_{ij} = c_{ijkl}\epsilon_{kl} \tag{174}$$

There are symmetries in the constant. Due to the conservation of angular momentum $\sigma_{ij} = \sigma_{ji}$, similarly $c_{ijkl} = c_{jikl}$. By definition strain is symmetric $\epsilon_{kl} = \epsilon_{lk}$, similarly $c_{ijkl} = c_{ijlk}$. Using strain energy definition $c_{ijkl} = \frac{\partial^2 W}{\epsilon_{ij}\epsilon_{kl}} = \frac{\partial^2 W}{\epsilon_{kl}\epsilon_{ij}}$, thus $c_{ijkl} = c_{klij}$. There are 21 independent constants in the linear stress strain relationship.

Plane of Symmetry: If a plane of symmetry occurs in a chosen coordinate system the strain in $\{x_1, x_2, x_3\}$ is the same as that in $\{-x_1, x_2, x_3\}$. The coordinate transformation between these two coordinate systems:

$$\widetilde{Q} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{175}$$

The transformation of the constant is as follows:

$$\bar{C}_{ijkl} = Q_{ai}Q_{bj}Q_{ck}Q_{dl}C_{abcd} \tag{176}$$

If there is a plane of symmetry $\bar{C}_{ijkl} = C_{ijkl}$. Thus: $C_{1112} = C_{1113} = C_{1222} = C_{1223} = C_{1233} = C_{1322} = C_{1323} = C_{1333} = 0$.The number of independent constants is reduced to 13.

Orthotropic: Using the transformations for the single degree of symmetry it can be shown that the 21 independent constants are reduced to 9.

Linear Isotropic: Let the strain energy function be defined only in terms of strain $W(\epsilon_{ij})$. In an isotropic system the energy does not depend on direction. Therefore the strain energy is a function of the invariants of the strain tensor $W(\mathbf{I}_\epsilon, \mathbf{II}_\epsilon, \mathbf{III}_\epsilon)$.

$$\sigma_{ij} = \frac{\partial W}{\partial \mathbf{I}_\epsilon}\frac{\mathrm{d}\mathbf{I}_\epsilon}{\mathrm{d}\epsilon_{ij}} + \frac{\partial W}{\partial \mathbf{II}_\epsilon}\frac{\mathrm{d}\mathbf{II}_\epsilon}{\mathrm{d}\epsilon_{ij}} + \frac{\partial W}{\partial \mathbf{III}_\epsilon}\frac{\mathrm{d}\mathbf{III}_\epsilon}{\mathrm{d}\epsilon_{ij}} \tag{177}$$

Recall one form of the invariant of a tensor: $\mathbf{I}_\epsilon = \epsilon_{kk}$; $\mathbf{II}_\epsilon = \frac{1}{2}\epsilon_{kl}\epsilon_{kl}$; $\mathbf{III}_\epsilon = \frac{1}{3}\epsilon_{kl}\epsilon_{lm}\epsilon_{mk}$. Take derivatives to construct:

$$\sigma_{ij} = \frac{\partial W}{\partial \epsilon_{kk}}\delta_{ij} + \frac{\partial W}{\partial \frac{1}{2}\epsilon_{kl}\epsilon_{kl}}\epsilon_{ij} + \frac{\partial W}{\partial \frac{1}{3}\epsilon_{kl}\epsilon_{lm}\epsilon_{mk}}\epsilon_{ik}\epsilon_{kj} = f_1\delta_{ij} + f_2\epsilon_{ij} + f_3\epsilon_{ik}\epsilon_{kj} \tag{178}$$

For a linear material there can't be any nonlinear terms. Thus, $f_3 = 0$. Similarly f_2 must be a constant. Therefore the strain energy function must be in the form $W = C\frac{1}{2}\epsilon_{kl}\epsilon_{kl} + D$. Where C and D are constants. Finally $f_1 = \frac{\partial \frac{1}{2}\epsilon_{kl}\epsilon_{kl}}{\partial \epsilon_{nn}} = \epsilon_{mm}$.

$$\widetilde{\sigma} = \lambda \ \mathrm{tr}(\widetilde{\epsilon})\widetilde{I} + 2\mu\widetilde{\epsilon} \tag{179}$$

A.6.4 Compatibility:

In a domain $\nabla \times \epsilon \times \nabla = 0$. In a multiply connected domain the line integrals about the openings must be zero. Given u_i calculating ϵ_{ij} results in a single value. Thus given ϵ_{ij}, u_i must be single valued. For two connected points on the domain an integral along the path connecting P_1 and P_2 must be path invariant.

$$\int_{P_1}^{P_2} \mathrm{d}\vec{u} = \vec{u}_{P_2} - \vec{u}_{P_1} = \int_{P_1}^{P_2} \mathrm{d}\vec{r} \cdot \nabla\vec{u} \tag{180}$$

Thus the integral around a closed path must be zero:

$$\int_{c_1} \mathrm{d}\vec{r} \cdot \nabla\vec{u} = \int_{c_2} \mathrm{d}\vec{r} \cdot \nabla\vec{u} \qquad \oint_{c_1+c_2} \mathrm{d}\vec{r} \cdot \nabla\vec{u} = 0 \tag{181}$$

Using Stokke's Theorem, for a singly connected domain

$$\oint \mathrm{d}\vec{r} \cdot \nabla\vec{u} = \int_S (\nabla \times \widetilde{\epsilon} \times \nabla)\mathrm{d}S = 0 \tag{182}$$

For a multiply connected domain, around each singularity:

$$\oint_{c_i} \mathrm{d}\vec{u} = 0 \tag{183}$$

The $\widetilde{\epsilon}$ is conserved in relation do displacements.

$$\begin{aligned} \widetilde{\epsilon} &= \frac{1}{2}(\nabla\vec{u} + \vec{u}\nabla) \\ \nabla \times \widetilde{\epsilon} &= \frac{1}{2}(\nabla \times \nabla\vec{u} + \nabla \times \vec{u}\nabla) \\ \nabla \times \widetilde{\epsilon} \times \nabla &= \frac{1}{2}(\nabla \times \vec{u}\nabla \times \nabla) = 0 \end{aligned} \tag{184}$$

A.6.5 Solvability Condition

According to Kirchhoff ([4], 160)

> If either the surface displacements or the surface traction are given, the solution for the problem of equilibrium of an elastic body...is unique in the sense that the state of stress (and strain) is determinate without ambiguity, provided that the magnitude of the stress (or strain) is so small that the strain energy function exists and remains positive definite.

The proof is by contradiction. Assume that two systems of displacements u_i' and u_i'' satisfy $\left(\frac{\partial W}{\partial e_{ij}}\right)_{,i} + \chi_i = 0$ — the equilibrium equation where χ_i is the effect of the body forces— and the boundary conditions: $S_u + S_\sigma$ defines the entire boundary surface; over S_u the values of u_i are given and over S_σ the traction $\vec{T}_i = \frac{\partial W}{\partial e_{ij}} v_j$ are specified. Then the difference $u_i' - u_i''$ satisfies the equation $\frac{\partial W}{\partial e_{ij}}{}_{,i} = 0$. Using Green's theorem and integrating over the volume $\int_V u_i \left(\frac{\partial W}{\partial e_{ij}}{}_{,i}\right) dV = 0$.

Appendix: Energy and Thermoelasticty ([7],41-46)

Kinetic Energy:

$$K = \frac{1}{2}\int_R \rho\vec{v}\cdot\vec{v}\mathrm{d}V \tag{185}$$

Change in Kinetic Energy:

$$\dot{K} = \frac{\mathrm{d}}{\mathrm{d}t}K = \int_R \rho\vec{v}\cdot\vec{a}\mathrm{d}V \tag{186}$$

External Power:

$$P = \int_R \rho\vec{b}\cdot\vec{v}\mathrm{d}V + \int_{\delta R}\vec{t}(\vec{n})\cdot\vec{v}\mathrm{d}S. \tag{187}$$

Using the definition of stress and the divergence theorem.

$$P = \int_R [(\sigma_{ij,j} + \rho b_i)v_i + \sigma_{ij}v_{i,j}]\mathrm{d}V \tag{188}$$

Thus the conservation laws can be constructed:

Conservation of Energy: Energy can neither be created or destroyed only changed in form. The change of kinetic energy and internal energy (e) in a control volume (cv) is equal to the work done on the surface by the traction ($\widetilde{P}$) and by the body forces ($\vec{b}$) and the heat convected into the volume and heat created by body($\vec{r}$) ([6]).

$$\begin{aligned}\frac{\mathrm{D}}{\mathrm{D}t}\int_{cv}(\rho\frac{v^2}{2}+\rho e)\ \mathrm{d}V = \int_s(\widetilde{P}\cdot\vec{n})\cdot\vec{v}\ \mathrm{d}S \\ +\int_{cv}\rho\vec{b}\cdot\vec{v}\ \mathrm{d}V - \int_s\vec{q}\cdot\vec{n}\ \mathrm{d}S + \int_{vc}\rho\vec{r}\end{aligned} \tag{189}$$

Using previous equations and methods:

$$\rho\frac{\mathrm{De}}{\mathrm{D}t} = \mathrm{tr}(\widetilde{\mathrm{P}}\nabla\vec{\mathrm{v}}) - \nabla\cdot\vec{\mathrm{q}} + \rho\vec{\mathrm{r}}. \tag{190}$$

First Law of Thermodynamics: Using conservation of energy: the system change in kinetic energy K and potential energy U is equal to the work done and the increase in energy in the system.

$$\dot{\bar{K}} + \dot{\bar{U}} = \bar{P} + \bar{Q} \tag{191}$$

Second Law of Thermodynamics: Given in cartesian coordinates:

$$\rho T\dot{\bar{S}} + q_{i,i} - \frac{1}{T}q_{i,i}T_{,i} - \rho r \geq 0 \tag{192}$$

S is the entropy and T is the temperature of the system. A process is *reversible* if the right hand side is equal to zero.

Appendix:Variational Calculus

([7],33–38)

C.1.1 Virtual Displacements:

Virtual Small Displacement Field: the difference between two neighboring kinematically admissible displacement fields $\delta\vec{u}$. The field is assumed to be infinitesimal: $|\delta u_{i,j}| << 1$.

Virtual Small Strain Field: $\delta\epsilon_{ij} = \frac{1}{2}(\delta u_{j,i} + \delta u_{i,j})$ The virtual displacement at any prescribed point, such as the boundary of a boundary value problem, is zero, $\delta\vec{u} = 0$ on δR_u.

Virtual Large Strain Field: as for real strain the large virtual strain is defined by square of the change in length.

$$\partial\epsilon_{ij} = l_0^2 + l^2 \tag{193}$$

C.2 Virtual Work:

Given a set of loads and a virtual displacement field $\delta\vec{u}$ the real loads acting on the virtual loads results in virtual work $\delta\bar{\mathbf{W}}$.

$$\delta\bar{\mathbf{W}}_{ext} = \int_R \rho b_i \delta u_i \mathrm{d}V + \int_{\delta R_i} t_i^a \delta u_i \mathrm{d}S \qquad \delta\bar{\mathbf{W}}_{int} = \int_R \sigma_{ij}\delta\epsilon_{ij}\mathrm{dV} \tag{194}$$

Since the small strain tensor (with zero assumed point force) is symmetric, and using the divergence theorem:

$$\delta\bar{\mathbf{W}}_{ext} - \delta\bar{\mathbf{W}}_{int} = \int_R (\delta\sigma_{ij,j} + \rho b_i)\delta u_i \mathrm{d}V - \int_{\delta R} (n_j\sigma_{ij} - t_i^a)\delta u_i \mathrm{d}S \tag{195}$$

Using the traction boundary conditions and the conservation of linear momentum (equilibrium):

$$\delta\bar{\mathbf{W}}_{ext} = \delta\bar{\mathbf{W}}_{int} \tag{196}$$

Principle of Virtual Work: A body is in static equilibrium if and only if the virtual internal work $\delta\bar{\mathbf{W}}_{int}$ is equal to the virtual external work $\delta\bar{\mathbf{W}}_{ext}$ and the traction boundary conditions are satisfied.

Virtual Stress Field: The difference between two statically admissible stress fields. $\delta\tilde{\sigma}$ can be constructed as for displacement.

$$\delta\sigma_{ij,j} = 0 \text{ in } R \text{ and } n_j\delta\sigma_{ij} = 0 \text{ on } \delta R_t \tag{197}$$

The external and internal complementary virtual work are defined respectively by:

$$\delta\bar{\mathbf{W}}^c_{ext} = \int_{\delta R_u} u_i^a n_j \delta\sigma_{ij}\mathrm{d}S \qquad \delta\bar{\mathbf{W}}^c_{int} = \int_R \epsilon_{ij}\delta\sigma_{ij}\mathrm{d}V \tag{198}$$

Again using the divergence theorem and the symmetry of strain the virtual work can be rewritten:

$$\delta\bar{\mathbf{W}}^c_{int} - \delta\bar{\mathbf{W}}^c_{ext} = \int_{\delta R_u} (u_i - u_i^a) n_j \delta\sigma_{ij}\mathrm{d}S + \int_R [\epsilon_{ij} - \frac{1}{2}(u_{i,j} + u_{j,i})]\delta\sigma_{ij}\mathrm{d}V \tag{199}$$

Principle of Virtual Forces (Principle of Complementary Virtual Work): work is conserved if and only if the strain yield $\tilde{\epsilon}$ is compatible with a kinematically admissible displacement field $\vec{u}$.

C.3 Internal Potential Energy Variations

Total Strain Energy: or Internal Potential Energy of the body.

$$\Pi_{int} = \int_R W \mathrm{d}V \tag{200}$$

Given the virtual displacement field $\delta\vec{u}$ and the corresponding virtual strain field $\delta\widetilde{\epsilon}$. the first variation of Π_{int} is $\delta\Pi_{int}$.

Let $\Pi_{int} + \Delta\Pi_{int}$ denote the internal potential energy evaluated at the perturbed displacement field $\vec{u} + \delta\vec{u}$. Assuming that the dependence of Π_{int} on $\vec{u}$ is smooth the taylor expansion can be used to describe the variation at δ.

$$\Delta\Pi_{int} = \delta\Pi_{int} + \frac{1}{2}\delta^2\Pi_{int} + \cdots \tag{201}$$

Thus the first variation is a linear approximation to $\Delta\Pi_{int}$. Also:

$$\begin{aligned}\Delta\Pi_{int} &= \int_R \left[W(\widetilde{\epsilon} + \delta\widetilde{\epsilon}) - W(\widetilde{\epsilon})\right] \mathrm{d}V \\ &= \int_R \frac{\partial W}{\partial\epsilon_{ij}}\delta\epsilon_{ij}\mathrm{d}V + \frac{1}{2}\int_R \frac{\partial^2 W}{\delta\epsilon_{ij}\partial\epsilon_{kl}}\delta\epsilon_{ij}\delta\epsilon_{kl}\mathrm{d}V + \cdots\end{aligned} \tag{202}$$

Compare terms, and use definition of virtual work, to find :

$$\delta\Pi_{int} = \delta\bar{W}_{int} = \int_R \sigma_{ij}\delta\epsilon_{ij}\mathrm{d}V \tag{203}$$

C.4 Total Potential Energy, Minimum principle

External Potential Energy: For a fixed set of loads $\vec{b}$ and $\vec{t}^a$.

$$\Pi_{ext} = -\int_R \rho(\vec{b}\cdot\vec{u})\mathrm{d}V - \int_{\partial R_t} \vec{t}^a\cdot\vec{u}\mathrm{d}S \tag{204}$$

Then the external work over the virtual displacement yield $\delta\vec{u}$ is

$$\delta\bar{W}_{ext} = -\delta\Pi_{ext}. \tag{205}$$

Define a local potential energy as $\Pi = \Pi_{int} + \Pi_{ext}$ thus $\delta\bar{W}_{int} - \delta\bar{W}_{ext} = \delta\Pi$. Using the principle of virtual work an elastic body is in equilibrium if and only if:

$$\delta\Pi = 0 \tag{206}$$

Furthermore, the equilibrium is stable if the total potential energy is a at a minimum. If Π is a local minimum, then for any nonzero virtual displacement $\delta\vec{u}$, $\Delta\Pi$ must be positive. Also, for small displacements $\delta^2\Pi_{ext} = 0$. Thus $\delta^2\Pi_{int} > 0$ and $\widetilde{\epsilon}^T\widetilde{\widetilde{C}}\widetilde{\epsilon}$ for all $\epsilon \neq 0$. This result may not be valid for displacement based energy derivations if the displacements can't be described by potentials. Also in such a case the second variation $\delta^2\Pi_{ext}$ is not necessarily zero.

C.5 Complementary Potential Energy

Complementary Potential Energy: for a given stress field $\widetilde{\sigma}$ and a set of prescribed displacements $\vec{u}_i^a$ independent of traction:

$$\Pi_{ext}^c = -\int_{\partial R} u_i^a n_j \sigma_{ij}\mathrm{d}S \quad \text{and} \quad \Pi_{int}^c = \int_R W^c(\widetilde{\sigma})\mathrm{d}V \tag{207}$$

Analogous to Total potential energy:

$$\delta \bar{W}^c_{int} - \delta \bar{W}^c_{ext} = \delta \Pi^c \tag{208}$$

Then the elastic system is at equilibrium if and only if the stress field $\widetilde{\sigma}$ is related to the stress-strain relations to a strain field that is compatible with both internal constraints and prescribed displacements or external constraints. Again Π^c is at a minimum at stable equilibrium.

C.6 The Calculus of Variations and Potential Energy ([?])

Consider a function with fixed endpoints $x(t_0) = x_0$ and $x(t_1) = x_1$ which is a piecewise smooth scalar function defined for all $t \in [t_0, t_1]$. There exists a scalar function of this function $x(\cdot)$, its derivative $\dot{x}(\cdot)$ and time t: $f[t, x, \dot{x}]$ which is also defined throughout the entire interval $t \in [t_0, t_1]$. It is convenient that this new function $f(\cdot)$ is continuous and contains as many partial derivatives as necessary. The **functional** $J(\cdot)$ is now the sum of $f(\cdot)$ over the range of t.

$$J(x(\cdot)) \hat{=} \int_{t_0}^{t_1} f[t, x(t), \dot{x}(t)] \mathrm{d}t \tag{209}$$

The **global absolute minimum** of $J(\cdot)$ occurs at x^* if and only if $J(x * (\cdot)) \leq J(x(\cdot)) \quad \forall x \in \{\text{domain}\}$. The **local minimum** of the integral occurs at x^* if within the immediate neighborhood of x^* the values of $J(\cdot)$ are greater than those at x^*. At a local minimum — a global minimum is also a local minimum – the rate of change of the functional is zero, the function is **stationary**.

Using various theorems of the calculus of variations presented in the book the Euler-Lagrangian form can be derived

$$f_{,x} = f_{,\dot{x}t} \dot{x}^*(t) + f_{,\dot{x}\dot{x}} \ddot{x}^*(t) \quad \forall t \in \mathbf{I} \tag{210}$$

Where $\mathbf{I}$ is the domain on which all the derivatives are continuous.

For mechanics problems the inverse function is of primary importance. If the function $x(t) = g(t, \alpha, \beta)$ is written in terms of a two parameter function, find the integrands $f(\cdot)$ which make the function $J(\cdot)$ stable. Assume that there exist continuous functions $\phi(\cdot)$ and $\psi(\cdot)$ to eliminate the constants.

$$\begin{aligned} x(t) &= g[t, \phi(t, x(t), \dot{x}(t)), \psi(t, x(t), \dot{x}(t))] \\ \dot{x}(t) &= g_t[t, \phi(t, x(t), \dot{x}(t)), \psi(t, x(t), \dot{x}(t))] \end{aligned} \tag{211}$$

Then $\ddot{x}(t) = G[t, x(t).\dot{x}(t)] = g_{tt}[t, \phi(t, x(t), \dot{x}(t)), \psi(t, x(t), \dot{x}(t))]$. And

$$f_{,x} - f_{,\dot{x}t} - \dot{x}(t) f_{,\dot{x}x} = G f_{,\dot{x}\dot{x}} \tag{212}$$

This solution must hold for every initial condition $\{x[t_0], \dot{x}[t_0]\}$.

Letting $\mathbf{M}(t, x, \dot{x}) \hat{=} f_{,\dot{x}\dot{x}}(t, x, \dot{x})$. And assuming that the derivative operation is linear $f_{,xr} = f_{,rx}$, then

$$\mathbf{M}_{,t} + \dot{x}\mathbf{M}_{,x} + G\mathbf{M}_{,\dot{x}} + G_{,\dot{x}}\mathbf{M} = 0 \tag{213}$$

The general solution to this partial differential equation is $\mathbf{M} = \frac{\mathbf{\Phi}}{\mathbf{\Theta}}$. Where $\mathbf{\Phi}$ is differentiable nonzero but otherwise arbitrary and

$$\mathbf{\Theta} \hat{=} \exp\left\{ \int G_{,\dot{x}} \left[t, g(t,\alpha,\beta), g_t(t,\alpha,\beta)\right] \mathrm{d}t \right\} \tag{214}$$

Finally functions $f(\cdot)$ can be found by integrating

$$f(t,x,\dot{x}) = \int_0^{\dot{x}} \int_0^{q} \mathbf{M}(t,x,p)\mathrm{d}p\mathrm{d}q + \dot{x}\lambda(t,x) + \mu(t,x) \tag{215}$$

Where $\lambda(\cdot)$ and $\mu(\cdot)$ are otherwise arbitrary functions which satisfy the continuity conditions and are defined on the required domain.

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