Advanced Mechanics of Fluids

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Introduction

Class meets in 545 Mudd monday from 12-2:30.
Grading: midterm .40 and final .60.
Homewoark is due mondays

- Outline:
  - Cartesian tensors (review)
  - Description of fluid motion – Kinematics
    - Lagrangian
    - Eulerian
  - Deformation and rate of strain
  - Conservation equations
  - Navier Stokkes equation
  - Vorticity
  - Potential flow
  - Laminar flows
  - Turbulence
- References:
  - Aries, R., *Vectors, Tensors, and the Basic Equations of Fluid Mechanics*
  - Chevray and Mathieu, J., *Topics in Fluid Mechanics*
  - Panton R., *Incompressible Flow*
  - White, F. W., *Viscous Fluid Flow*

1 Cartesian Tensors (Review)

1.1 Vectors

Vector: a vector can be represented by its components in a n dimensional coordinate system.

$$\vec{a} = \{a_1, a_2, a_3 \ldots\} \tag{1}$$

vectors are equal if the components are equal in the same coordinate system.

Null Vector: all components are zero.

Negative Vector:

$$\vec{b} = -\vec{a} \qquad b_i = a_i \tag{2}$$

Scalar Product (dot):

$$\vec{a} \cdot \vec{b} = a_i b_i \tag{3}$$

Vector Product:

$$\vec{a} \times \vec{b} = \epsilon_{ijk} a_i b_j \vec{e}_k \tag{4}$$

1.1.1 Transformation of Vectors

$$\vec{a} = a_i \vec{\theta}^i = a'_i \vec{\theta}'^i \tag{5}$$

$$\frac{\partial}{\partial \vec{\theta}_i}(\vec{a}_i \vec{\theta}^i) = \frac{\partial}{\partial \vec{\theta}^i}(\vec{a}'_i \vec{\theta}'^i) \tag{6}$$

$$\vec{a}_i = \vec{a}'_i \frac{\partial \vec{\theta}'_i}{\partial \vec{\theta}_i} \tag{7}$$

For cartesian tensors the cosine of the angle between the two cartesian coordinates is the partial derivative one direction vector relative to the other. The transformation is orthogonal so covarient and contravarient tensors are equal.

If a quantity transforms as the vector above, see more in qualifying exam notes section tensor, then it is a tensor. Also see definitions of the kronecker delta and the alternating tensor.

1.1.2 Tensor Operations

The sum, addition, scalar product, cross product, triple scalar product and triple vector product of tensors is also a tensor.

1.1.3 Vector Calculus

Gradient:

$$\nabla f = \frac{\partial f}{\partial x_i} \vec{g}_i \tag{8}$$

$$\nabla \vec{a} = \frac{\partial a_j}{\partial x_i} \vec{g}_i \tag{9}$$

Divergence:

$$\nabla \cdot \vec{a} = \frac{\partial a_i}{\partial x_i} \tag{10}$$

Curl:

$$\nabla \times \vec{a} = \epsilon_{ijk} \frac{\partial u_j}{\partial x_i} \vec{g}_k \tag{11}$$

2 Kinematics

2.1 Description of Fluid Motion

Lagrangian: description of fluid motion constructed by following individual particles *particle point of view* — Spatial description.

$$\vec{r} = \vec{f^l}(\vec{x}^0, t) \tag{12}$$

Eulerian: description of fluid motion constructed by observing the passage of particles through a fixed position in space the *field point of view* — Material description.

$$\vec{r} = \vec{f^e}(\vec{x}, t) \tag{13}$$

There is a correspondence between the Lagrangian and Eulerian description. The particle which arrives in the Eulerian observed field at time t can be described by its initial condition $\vec{x}^0$ (or marker or color) using the lagrangian description: $x_i = f_i^l(\vec{x}^0, t)$. The field position is now described in terms of the markers of the particle and the eulerian description can be formulated in the lagrangian perspective: $x_i = f_i^e(f_1^l(\vec{x}, t), f_2^l(\vec{x}, t), f_3^l(\vec{x}, t), t)$. The lagrangian can be solved using an inverse procedure.

2.2 Steady and Unsteady flow

Steady Motion: If at various points of flow all quantities (velocity, density ...) associated with thte flow remain unchanged with time.

Unsteady Motion: not steady motion.

2.3 Velocities

The velocity of fluid described from a lagrangian perspective.

$$v_i = \frac{\partial x_i}{\partial t} \tag{14}$$

The *streamline* is parallel to the velocity field at every point at time $t = t_1$.

$$\mathrm{d}\vec{r} \times \vec{v} = 0,\ E_{ijk} \mathrm{d}r_i v_j \vec{e}_k = 0 : \quad (w\mathrm{d}y - v\mathrm{d}z) = (u\mathrm{d}z - w\mathrm{d}x) = (v\mathrm{d}x - u\mathrm{d}y) = 0 \tag{15}$$

$$\frac{u}{\mathrm{d}x} = \frac{v}{\mathrm{d}y} = \frac{w}{\mathrm{d}z} \tag{16}$$

2.3.1 Local, Convective and Material Derivatives

Look at the eulerian description of flow ϕ at time t and time $t + \Delta t$.

$$\frac{\Delta \Phi}{\Delta t} = \frac{\phi(\vec{r} + \vec{v}\Delta t, t + \Delta t) - \phi(\vec{r}, t)}{\Delta t} \tag{17}$$

$$\frac{\Phi(\vec{r} + \vec{v}\Delta t, t + \Delta t) - \Phi(\vec{r} + \vec{v}\Delta t, t)}{\Delta t} + \frac{\phi(\vec{r} + \vec{v}\Delta t, t) - \phi(\vec{r}, t)}{\Delta t} \tag{18}$$

Take the limit as $\Delta t- > 0$ to finde the *substantive* or *stokkes* or *material* or *total derivative* as the sum of the *local derivative* and the *convective derivative.*

$$\frac{D\phi}{Dt} = \frac{\partial \phi}{\partial t} + \vec{v} \cdot \nabla \phi \tag{19}$$

Notice, that for steady flow the *convective derivative* is not automatically zero, acceleration can occur in a steady flow.

2.3.2 Characteristic Lines

Streamline: at time instant (t_i) the line is parallel to the velocity field $\vec{v}(t_1)$.

Pathline: line described one particle as time passes (trajectory).

Streakline: a line which joines all particles at $t = t_1$ which passes through a given point at previous time $t \leq t_1$.

Timeline: a line which connects particles at time t_1 that passed through a line at time t_0.

See homework for example.

3 Deformation and Rate of Strain

Experimental result show that fluid stress depends on the local rate of strain. The taylor expansion of the velocity:

$$U_i' = U_i + \frac{\partial U_i}{\partial x_j}\delta x_j + \frac{\partial^2 U_i}{\partial x_k \partial x_l}\frac{\delta x_k \delta x_l}{2!} + \cdots \tag{20}$$

Use only the first order term:

$$\vec{U}' - \vec{U} = \nabla \vec{U} \cdot \delta \vec{x} \tag{21}$$

The rate of deformation tensor: $\widetilde{D} = \nabla U$. Divide the second order tensor into its symmetric and unsymmetric parts.

$$\widetilde{D} = \frac{1}{2}(\widetilde{D} + \widetilde{D}^T) + \frac{1}{2}(\widetilde{D} - \widetilde{D}^T) \tag{22}$$

The rate of strain tensor: is the symmetric part: $\widetilde{E} = \nabla \vec{U} + \vec{U}\nabla$.

Vorticity: is related to the unsymmetric part: $\frac{1}{2}\widetilde{\Omega} = \frac{1}{2}(\nabla U - U\nabla)$, the voricity is a vector: $\Omega_k = \epsilon_{ijk}\Omega_{ij}$.

3.1 Rate of Strain

The *translation* does not cause any deformation of the flowing fluid particle. *Rotation* also doesn't cause any deformation of the flowing fluid. *Linear deformation* is an increase or decrease in fluid shape. If the density of the element remains constant linear deformation is a function of the conservation of mass $\nabla \cdot \vec{U} = 0$ — isopychnic flow. *Angular deformation* is not a function of the conservation of mass.

Quadric: A second degree surface in three dimensions is associated with each point of the equation: $\widetilde{E} : (\vec{x} \otimes \vec{x}) = C_{onst}$.

The relative values of the normal strain rates (E_{11}, E_{22}, E_{33}) determine if the quadric is a paraboloid, ellipsoid or hyperbloid — Conics. The principal strain rates direction can be fiewed as the axis of symmetry of the quadric. It is the direction in which angles do not change.

3.2 Transformation of Coordinates

The orthonormal transformation $\widetilde{Q}$(no change in lengths or angles between vectors transformed $\widetilde{Q}^T\widetilde{Q} = \widetilde{I}$.)

$$\frac{1}{\delta}\frac{\mathrm{d}\delta}{\mathrm{d}t} = \frac{1}{2}\left[E_{11}\cos^2\alpha + E_{22}\sin^2\alpha + 2E_{12}\sin\alpha\cos\alpha\right] \tag{23}$$

$$\mathrm{d}\alpha = \frac{1}{\delta}(\mathrm{d}y\cos\alpha - \mathrm{d}x\sin\alpha); \quad \frac{\mathrm{d}\alpha}{\mathrm{d}t} = \frac{1}{4}(E_{22} - E_{11})\sin^2\alpha \tag{24}$$

$$\frac{\mathrm{d}\alpha}{\mathrm{d}t} = \frac{1}{4}\left[(E_{22} - E_{11})\sin 2\alpha + 2E_{12}\cos 2\alpha)\right] - \frac{1}{2}\Omega_{12} \tag{25}$$

3.3 Principal Direction

For no vorticity there is no angle change in the principal direction.

$$0 = (E_{22} - E_{11})\sin 2\alpha_p + 2E_{12}\cos 2\alpha_p; \quad \alpha_p = \frac{1}{2}\tan^{-1}\left(\frac{-2E_{12}}{E_{22} - E_{11}}\right) \tag{26}$$

3.4 Examples

3.4.1 Linear, Isophychnic Flow

Given the flow profile $\vec{U} = \vec{f}(\vec{x})$, $\vec{V} = 0$, and $\nabla \cdot \vec{U} = 0$. Thus $E_{22} = E_{11} = 0$. The principal angle is: $\alpha_p = \frac{1}{2}\tan^{-1}\frac{E_{12}}{E_{11}} = \frac{\pi}{4}, \frac{3\pi}{4}, \ldots$. Notice that the principal angle is invarient of $\vec{f}(\vec{x})$.

3.4.2 Polar Coordinate Formulation

$$x = r\cos\theta \quad y = r\sin\theta \tag{27}$$

The flow is given as: $\vec{U} = \vec{q}_r \cos\theta - \vec{q}_\theta \sin\theta$ and $\vec{V} = \vec{q}_r \sin\theta + \vec{q}_\theta \cos\theta$. The vorticity is given as follows $\Omega = \frac{1}{r}\frac{\partial q_r}{\partial \theta} + \frac{1}{r}\frac{\partial}{\partial r}(rq_\theta)$. The corresponding strain rate components are: $D_{rr} = 2\frac{\partial q_r}{\partial r}$, $D_{\theta\theta} = 2\left(\frac{1}{r}\frac{\partial q_\theta}{\partial \theta} + \frac{q_r}{r}\right)$, and $D_{r\theta} = \left(\frac{1}{r}\frac{\partial q_r}{\partial \theta} + r\frac{\partial}{\partial r}\frac{q_\theta}{r}\right)$.

3.4.3 Rectilinear motion with vorticity

3.4.4 Flow in concentric circles

3.4.5 Hurricane

4 Conservation Equations

Derivend in the Eulerian fram of reference. Study the flow through acontrol volume V bounded by a control surface S. Any extensive property $M(\vec{x})$ can be given intensively- – per unit mass —. In general the time rate increase of M inside S is the net rate of influx through S or from the surface sources on S plus the net rate of inflow from sources within V.

$$\frac{\partial}{\partial t}\int_V M \mathrm{d}V = \int_S \Sigma_M \mathrm{d}S + \int_V \sigma_M \mathrm{d}V \tag{28}$$

Here Σ_M and σ_M are associated with some physical phenomena. Since Σ_M is defined on a surface it can be written as $\vec{n}\cdot\vec{\mu}$ where $\vec{n}$ is the outward unit normal to the surface and $\vec{\mu}$ is the vector field related to the intrinsic property of M. Applying the Divergence Theorem

$$\int_V \left\{\frac{\partial M}{\partial t} - \nabla\cdot\vec{\mu} - \sigma_M\right\}\mathrm{d}V = 0. \tag{29}$$

Since V is consistent over time and arbitrary:

$$\frac{\partial M}{\partial t} - \nabla\cdot\vec{\mu} - \sigma_M = 0 \tag{30}$$

Given a time dependent extensive property $M(\vec{x},t)$ the surface components Σ_M can be defined as the sum $-\vec{n}\cdot\vec{\mu}M + \Sigma'_M$. The $\vec{\mu}$ is defined as $\vec{\mu} = -\vec{\mu}M + \vec{\mu}'$. Examples of M include mass and momentum. Quantities which are not transported include radiation.

$$\int_V \left[\frac{\partial M}{\partial t} + \nabla\cdot(\vec{\mu}M) - \nabla\cdot\vec{\mu}' - \sigma_M\right]\mathrm{d}V \tag{31}$$

Again using the control volume argument:

$$\frac{\partial M}{\partial t} + \vec{u}\cdot\nabla M = \frac{DM}{Dt} = -M\nabla\cdot\vec{\mu} + \nabla\cdot\vec{\mu}' + \sigma_M. \tag{32}$$

One application of this formulation is linear momentum. If M is a component of Momentum then $\nabla \cdot \mu'$ is a surface force due to viscosity or pressure gradient and σ_M is a body force.

The Eulerian formulation above is related to the Lagrangian formulation.

System: A system consists of a definite number of particles with a definite mass and it is distinguished from all other matter — its sorroundings.

Control Volume: A control volume is a fixed region in space.

The change of a physical quantity M in time can be described in terms of either system.

$$\left(\frac{\partial M}{\partial t}\right)_{\text{system}} = \lim_{\Delta t \to 0} \frac{M_{t+\Delta t} - M_t}{\Delta t} \tag{33}$$

From above

$$\left(\frac{DM}{Dt}\right)_{\text{material}} = -M\nabla \cdot \vec{\mu} + \nabla \cdot \vec{\mu}' + \sigma_M. \tag{34}$$

If the control volume and system are at the same place at time t_0. Then after a short time Δt. If Δt is short enough then the overlapping volume is almost the same as the first initial control volume. The change of the physical quantity M in a system can be written as:

$$\begin{aligned}
\left(\frac{\partial M}{\partial t}\right)_{\text{system}} &= \lim_{\Delta t \to 0} \frac{(\text{material at time } t + \Delta t) - (\text{material at time } t)}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{(\text{overlap} + \text{not c.v.}_t)_{t+\Delta t} - (\text{not system} + \text{overlap})_t}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{(\text{overlap})_{t+\Delta t} - (\text{overlap})_t}{\Delta t} \\
&+ \lim_{\Delta t \to 0} \frac{(\text{not c.v.}_t)_{t+\Delta t} - (\text{not system})_t}{\Delta t}
\end{aligned} \tag{35}$$

The material in the volumes *not system* and *not c.v.* is the material which has traveld in and out of the control volume at time t. As δt approaches zero the overlap region is almost the same as the control volume. Thus the change of mass within the system can be written as:

$$\left(\frac{\partial M}{\partial t}\right)_{\text{system}} = \frac{\partial}{\partial t}\int_{c.v.} \mu\rho \mathrm{d}V + \int_{c.s.} \mu\rho\vec{v} \cdot \mathrm{d}\vec{A}. \tag{36}$$

The Eulerian description of the conservation change in mass is instead written:

$$\frac{\partial \rho}{\partial t} = -\rho\nabla \cdot \vec{u} - \vec{u} \cdot \nabla\rho \tag{37}$$

The result can also be found using Leibniz rule:

$$\frac{\partial}{\partial x}\int_{a(x)}^{b(x)} f(x,y)\mathrm{d}y = \int_{a(x)}^{b(x)} \frac{\partial f}{\partial x}\mathrm{d}y + f[x,b(x)]\frac{\mathrm{d}b}{\mathrm{d}x} - f[x,a(x)]\frac{\mathrm{d}a}{\mathrm{d}x} \tag{38}$$

Using this the result can be found automatically:

$$\frac{\partial}{\partial t}\int_V M\mathrm{d}V = \int_V \frac{\partial M}{\partial t}\mathrm{d}V + \int_S M\vec{u}\cdot\vec{n}\mathrm{d}S \tag{39}$$

Using the divergence theorem, and recalling that V is an arbitrary control volume.

$$\frac{\partial M}{\partial t} + \nabla\cdot(\vec{u}M) = \nabla\cdot\vec{\mu}' + \sigma_M \tag{40}$$

Also, the material derivative of an integral over the control volume is given as:

$$\frac{\mathrm{d}}{\mathrm{dt}}\int_V M\mathrm{d}V = \int_V \left\{\frac{\partial M}{\partial t}\mathrm{d}V + M\frac{\partial}{\partial t}\mathrm{d}V\right\} \tag{41}$$

For a small spatial volume τ bounded by D:

$$\nabla\cdot\vec{g} = \lim_{\tau\to 0}\left\{\frac{1}{\tau}\int_D \vec{n}\cdot\vec{g}\mathrm{d}D\right\} \tag{42}$$

If τ is $\mathrm{d}V$ then as $\tau \to 0$

$$\nabla\cdot\vec{u} = \frac{1}{\mathrm{d}V}\int_S \vec{n}\cdot\vec{u}\mathrm{d}S \quad \text{and} \quad \frac{\partial}{\partial t}\mathrm{d}V = (\nabla\cdot\vec{u})\mathrm{dV} \tag{43}$$

5 Equations of Motion

5.1 Navier Stokes Equation

Derive the equation by combining the constitutive relation for a linear isotropic fluid – Newtonian flud, and the linear momentum relation. The stress in the fluid $\widetilde{S}$ related to the strain rate $\widetilde{E}$:

$$\widetilde{\widetilde{E}} = \frac{1}{2}(\nabla\vec{u} + \vec{u}\nabla) \quad \text{and} \quad \widetilde{S} = \widetilde{S}^0 + \widetilde{\widetilde{C}} : \widetilde{E} \tag{44}$$

Combinig to find

$$\widetilde{S} = \widetilde{S}^0 + 2\mu\widetilde{E} + \lambda\mathrm{tr}(\widetilde{E})\widetilde{I} \tag{45}$$

Putting this result together with the conservation of linear momentum, for constant viscosity :

$$\rho\frac{\mathrm{Du}}{\mathrm{Dt}} = \rho\vec{\chi} - \nabla P + \mu\nabla^2\vec{u} + \frac{\mu}{3}\nabla(\nabla\cdot\vec{u}) \tag{46}$$

Euler's equation is for inviscid, incompressible flow

$$\frac{\mathrm{Du}}{\mathrm{D}t} = \vec{\chi} - \frac{1}{\rho}\nabla P \tag{47}$$

5.2 Shear Driven Flow: Couette Flow

Solve Navier stokes equation for incompressible unidirectional flow to find that the velocity ditribution is linear, 0 at the fixed wall and U_0 at the moving wall. In such a flow the presure is hydrostatic.

5.3 Pressure Driven Flow: Poisille Flow:

Given a constant pressure gradient k, a viscosity ν, and a density ρ, the parabolic flow profile is proportinal by $-\frac{k}{2\mu}$.
The flows can be combined linearly.

Example: 2D flow of different fluids down an inclined plane, an established flow. No vertical flow – if there were flow would not be established. *Boundary conditions*: Zero traction at interface of fluids. Zero change in flow at the surface.

Separation of Variables

See PDE

5.4 Stokke's first problem

Flow between two fixed plates. The governing equation of flow is

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}. \tag{48}$$

The boundary conditions are $u(t, y = 0) = U_0$ and $u(t, y = h) = 0$. An initial condition must also be given.

Example: Rocker Bearing is a combination of Couette and Poiseille flow.

6 Vorticity

For isopychnic flow

$$\Omega = \frac{\partial V}{\partial x} - \frac{\partial U}{\partial y}. \tag{49}$$

In general vortex flow is

$$\vec{\Omega} = \nabla \times \mathrm{d}\vec{r} \quad \text{and} \quad (\nabla \times \vec{v}) \times \mathrm{d}\vec{r} = 0. \tag{50}$$

The vorticity then is

$$\vec{\Omega} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = \xi\vec{i} + \eta\vec{j} + \zeta\vec{k} \tag{51}$$

6.1 Circulation

By mass flux analogy consider a surface inside a contour c.

$$\Gamma = \int_A \vec{\Omega} \cdot \mathrm{d}A = \int_A (\nabla \times \vec{v}) \cdot \mathrm{d}A = \oint_S \vec{v} \cdot \mathrm{d}S \tag{52}$$

6.1.1 Flow in Concentric Circles

$$\Gamma(r) = \oint_c \vec{v}_\theta(r) \cdot R \mathrm{d}\theta = 2\pi r \vec{v}_\theta(R) \quad \text{and} \quad V_\theta = \frac{A}{r} \tag{53}$$

$$\Gamma(r) = 2\pi R \frac{A}{R} = 2\pi A \quad \text{is constant} \tag{54}$$

For any contour not including the center the circulation is zero, no vorticity. For any simply connected contour including the center the circulation is constant.

Vortex lines move with the fluid. a vortex tube cannot end within the fluid it must end at a solid boundary or end on itself. On a given vortex lne the quantity $\frac{w}{\rho l}$ is constant, where w is the vorticity l is hte lenght of the line, and ρ is density.

6.1.2 Rigid body Rotation

$$\Omega = \Omega_0 \quad \text{is constant} \tag{55}$$

$$V_\theta = \frac{\Omega_0}{2} r \tag{56}$$

6.1.3 Helmholz Theorem

The circulation around a vortex tube must be the same at all cross sections. The kinematic result is valid for all viscid and inviscid flows.

Vortex tubes and stream tubes do not conincide in general except for Beltrami Flows. Vortex tubes are similar to stream lines in that they are parallel to voricity like stream lines are parallel to velocity.

6.1.4 Kelvin's Circulation Theorem

$$\frac{\mathrm{D}\Gamma}{\mathrm{D}t} = -\oint \frac{\mathrm{D}P}{\rho} + \oint \vec{G} \cdot \mathrm{d}\vec{r} + \oint \nu \nabla^2 \vec{v} \cdot \mathrm{d}\vec{r} \tag{57}$$

6.1.5 Conservation of Angular momentum

The navier stokkes equations can be rewritten such that

$$\frac{\mathrm{D}\vec{w}}{\mathrm{D}t} = \vec{w} \cdot \nabla \vec{u} + \nu \nabla^2 \vec{w} \tag{58}$$

6.2 Circulation along a line

The rate at wich flow along a line is increasing is a function of the rate at which the neds of hte elemtn are drawing apart – linear deformation:

$$\frac{\mathrm{D}(u\delta x)}{\mathrm{D}t} = \frac{\mathrm{D}u}{\mathrm{D}t}\delta x + u\frac{\mathrm{D}(\delta x)}{\mathrm{D}t}. \tag{59}$$

The three Navier Stokke's equations can be formulated as follows to inculde circulation if the body forces are derivable from potential.

$$\frac{\mathrm{D}}{\mathrm{D}t}(u\delta x + v\delta y + w\delta z) = -\delta\Omega - \frac{\delta P}{\rho} + u\delta u + v\delta v + w\delta w \tag{60}$$

Using the baratropic assumption that ρ depends only on P.

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_a^b (u\delta x + v\delta y + w\delta z) = \left[-\Omega - \int \frac{\delta P}{\rho} + \frac{\vec{v}^2}{2}\right]_b^a \tag{61}$$

The rate at which the line integral increases depends only onthe endpoints, thus for a closed loop flow $a = b$

$$\frac{\mathrm{D}}{\mathrm{D}t}\oint (u\delta x + v\delta y + w\delta z) = 0 \tag{62}$$

The circulation Γ is defined as $\oint(u\delta x + v\delta y + w\delta z)$. Thus the circulation around any closed loop moving with the fluid will remain constant.

Also, in the absence of streses due to the deformation the vorticity around any fluid element can not change

7 Inviscid Flow and Conservative Systems Analysis

7.1 Potential Flow Φ

A method for solving flow calculations is in closed form. Assumptions required include that the applied forces are conservative, no dissipation. That is there are no viscous effects. Consequently the change in force potential is equal to teh work done by conservative forces (the sign change is a matter of convention):

$$\Phi_a - \Phi_b = \int_a^b \vec{F} \cdot \mathrm{d}\vec{s} \tag{63}$$

Thus, $\vec{F}$ as a the gradient of the potential $\vec{F} = -\nabla\theta$. In cartesian coordinates:

$$u = -\frac{\partial\theta}{\partial x} \quad \text{and} \quad v = -\frac{\partial\theta}{\partial y} \tag{64}$$

7.1.1 Irrotational flow

$$2w \hat{=} \nabla \times \vec{v} = 0 \tag{65}$$

In terms of potential flow $\nabla \times (-\nabla\phi)$ is automatically zero. Thus potential flow is irrotational. A boundary layer is not irrotational. The flow outside a rankine vortex is irrotational, inside the vortex is rotational.

7.1.2 Euler's Equation

The navier stokke's equation for incompressible flow and only gravity body forces can be rewritten as:

$$\frac{\mathrm{D}\vec{v}}{\mathrm{D}t} = -\frac{1}{\rho}\nabla\left(P + \gamma h\right) \tag{66}$$

In terms of potential Φ:

$$\frac{\partial(\nabla\Phi)}{\partial t} + (\nabla\Phi)\cdot\nabla(\nabla\Phi) = -\frac{1}{\rho}\nabla\left(P + \gamma h\right) \tag{67}$$

$$\nabla\left[\frac{\partial\Phi}{\partial t} + \frac{1}{2}(\vec{v}\cdot\vec{v}) + \frac{P}{\rho} + gh\right] = 0 \tag{68}$$

For convenience set $q = \vec{u}\cdot\vec{u}$. Thus the flow is not a function of spatial coordinates.

$$\frac{q^2}{2} + \frac{P}{\rho} + gh - \frac{\partial\Phi}{\partial t} = F(t) \tag{69}$$

At a given time t the sum is constant everywhere, not just along streamlines as it is in Bernouli's equation.

7.2 The Stream Function Ψ

The flow rate at between a fixed point A and a variable point $P(x, y)$ is given by $P \hat{=} \Psi(x, y)$.

Streamline: along stream lines the stream function Ψ is constant, the flowrate is constant since streamlines are defined to be parallel to the flow velocity and using continuity.

By calculating the flow between two streamlines which are infinitely far apart teh velocity can be defined in cartesian coordinates as:

$$v = \frac{\partial\Psi}{\partial x} \quad \text{and} \quad u = -\frac{\partial\Psi}{\partial y}. \tag{70}$$

And in polar coordinates as:

$$V_r = \frac{-1}{r}\frac{\partial\Psi}{\partial\theta} \quad \text{and} \quad V_\theta = \frac{\partial\Psi}{\partial r}. \tag{71}$$

Now notice that the potential and the stream function are related by the Cauchy-Riemann Equations:

$$\frac{\partial\theta}{\partial x} = \frac{\partial\Psi}{\partial y} \quad \text{and} \quad \frac{\partial\theta}{\partial y} = -\frac{\partial\Psi}{\partial x}. \tag{72}$$

7.2.1 Boundary Conditions

Here $\vec{q}$ is the velocity of the fluid:

- For fixed boundaries, the velocity component normal to the boundary is zero at every point on the boundary. $\vec{q} \cdot \vec{n} = 0$ if $\vec{n}$ is the unit outward normal from the boundary, similarly $\frac{\mathrm{d}\Phi}{\mathrm{d}n} = 0$. There is no shear resistance at the boundary since the viscosity everywhere is by definition zero.
- For boundaries moving at velocity v the normal components of the velocities of the fluid and the wall must be equall.

$$\vec{q} \cdot \vec{n} = \vec{v} \cdot \vec{n} \quad \text{and} \quad (\vec{q} - \vec{v}) \cdot \vec{n} = 0. \tag{73}$$

- At the interface between two fluids velocity differences may exist since there is no viscosity but the pressure must be continuous across the interface.

8 Conformal Mapping

9 Low Reynolds Number Flow

10 Turbulence

10.1 Boundary Layer Turbulence

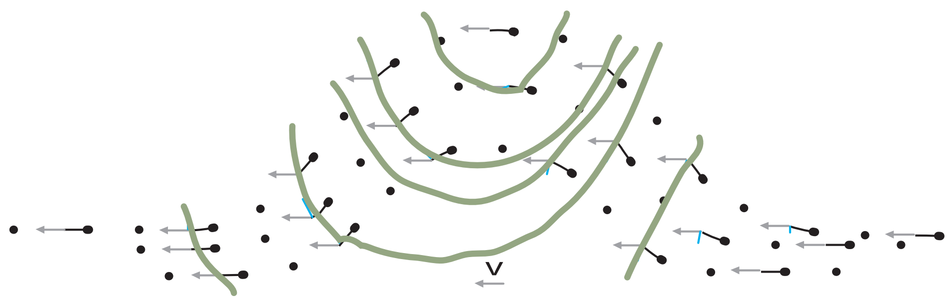
E6100

Homework 2

Elisabeth Malsch

9/13/2000

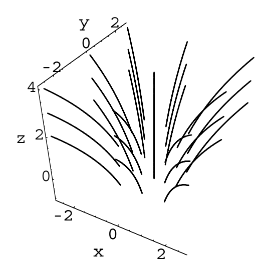
1. **Show the graphical constructions of streamlines for the unsteady wave:**



2. **Find the streamline, pathline and streakline for the flow equations given by $u = x/t,\ v = y,\ w = 3$.**

Streamline: at time instant (t_1) the stremline is parallel to the velocity field $\vec{v}(t_1)$.

$$\frac{u}{dx} = \frac{v}{dy} = \frac{w}{dz} = \frac{1}{ds} \tag{1}$$



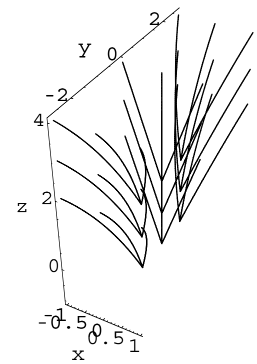
$$\frac{dx}{ds} = \frac{x}{t}, \quad x = Ae^{s/t} \tag{2}$$

$$\frac{dy}{ds} = y, \quad y = Be^{s} \tag{3}$$

$$\frac{dz}{ds} = 3, \quad z = 3s + C \tag{4}$$

$\{A, B, C\}$ are the starting points of the streamlines. (Plot is for various starting points at t=1)

Pathline: line which describes the path of one particle as time passes.



$$u = \frac{dx}{dt} = \frac{x}{t}, \quad x = At \quad (5)$$

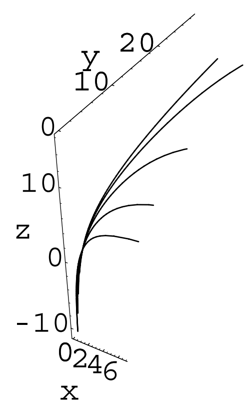
$$v = \frac{dy}{dt} = y, \quad y = Be^{t} \quad (6)$$

$$w = \frac{dz}{dt} = 3, \quad z = 3t + C \quad (7)$$

$\{A, B, C\}$ descries the particles position at t_0.

Streakline: a line which joines all particcles at $t = t_1$ which passed through a given point at a previous time $t < t_1$.

Location particles pass: x', y', z'. Solving the pathline equations for the appropriate constants as a function of τ where $t_0 \leq \tau \leq t_1$:



$$a' = \frac{x'}{\tau}, \quad x = \frac{x'}{\tau} t \quad (8)$$

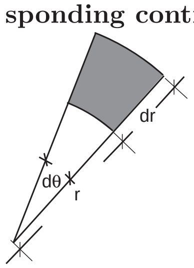
$$b' = y' e^{-\tau}, \quad y = y' e^{t-\tau} \quad (9)$$

$$c' = z' - 3\tau, \quad z = 3(t - \tau) + z' \quad (10)$$

(Plot is given for streaklines starting at point 1,1,1 at times $t1 = 1, 2, 3, 4$ and 5.)

Elisabeth Malsch
Homework 4
9/28/2000

1. **Define an appropriate fluid element in cylindrical coordinates (r, θ, z) and use the conservation of mass to obtain the corresponding continuity equation.**

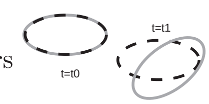


$$\begin{aligned} 0 &= \frac{\partial}{\partial t}\int_{c.v.} \rho \mathrm{d}V + \int_{c.s.} \rho \vec{v}\cdot \mathrm{d}\vec{A} \\ &= \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) \end{aligned} \tag{1}$$

$$\begin{aligned} 0 &= \frac{\partial}{\partial t}\left[\rho \left(r^2 - (r+\mathrm{d}r)^2\right)\frac{\mathrm{d}\theta}{2}\right] - \rho(v_r r \mathrm{d}\theta \mathrm{d}z) \\ &\quad +(\rho + \frac{\partial \rho}{\partial r}\mathrm{d}r)(v_r + \frac{\partial v_r}{\partial r}\mathrm{d}r)(r+\mathrm{d}r)\mathrm{d}\theta\mathrm{d}z - \rho(v_z r \mathrm{d}r\mathrm{d}\theta) \\ &\quad +(\rho + \frac{\partial \rho}{\partial z}\mathrm{d}z)(v_z + \frac{\partial v_z}{\partial z}\mathrm{d}z) r\mathrm{d}r\mathrm{d}\theta - \rho(v_\theta \mathrm{d}r\mathrm{d}z) \\ &\quad +(\rho + \frac{\partial \rho}{\partial \theta}\mathrm{d}\theta)(v_\theta + \frac{\partial v_\theta}{\partial \theta}\mathrm{d}\theta) \\ &= \frac{\partial}{\partial t}\rho + v_r\frac{\partial \rho}{\partial r} + \frac{v_\theta}{r}\frac{\partial \rho}{\partial \theta} + v_z\frac{\partial \rho}{\partial z} + \rho\left[\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z}\right] \end{aligned} \tag{2}$$

2. **The mass flux through a closed surface within a fluid is found to be different from zero. Which situations would lead to this result?**

When the closed surface is moving the mass flux appears to be something other than zero.



3. **For which values of w would the following steady field satisfy the continuity equations for an incompressible fluid.**

$$u = \ln\left(\frac{y^2}{b^2} + \frac{z^2}{c^2}\right), \quad v = \sin\left(\frac{x^2}{a^2} + \frac{z^2}{c^2}\right), \quad w = f(x,y,z) \tag{3}$$

The continuity equation for incompressible flow is $\nabla \cdot \vec{u} = 0$. Thus $u_{,x} + v_{,y} + w_{,z} = 0$.

$$\begin{aligned} \frac{\partial u}{\partial x} &= 0 \\ \frac{\partial v}{\partial y} &= 0 \end{aligned} \tag{4}$$

$$w_{,z} = 0 \tag{5}$$

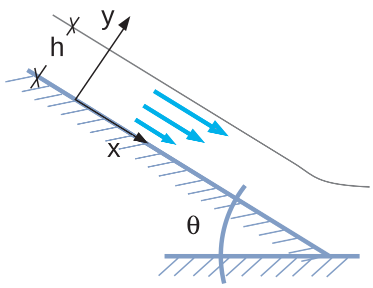
Thus, w can only be a function of x or y and not a funciton of z.

Elisabeth Malsch
Homework 3
9/28/2000

1. **The velocity profile for an established flow down an inclined plane making an angle θ with the horizontal**

$$u = Ay(2h - y) \quad v = 0 \quad w = 0 \tag{1}$$

Find A as a function of γ, μ, and θ, and the time rate of change of volume (discharge Q).



Conservation of Mass ($\nabla \cdot \vec{v} = 0$):

$$A_0 V_0 = A_1 V_1 = Q \tag{2}$$

Conservation of Momentum (with given conditions):

$$\begin{aligned} 0 &= \chi_x + \mu \frac{\mathrm{d}^2 u}{\mathrm{d}y^2} \\ 0 &= \chi_y + \frac{\partial P}{\partial y} \end{aligned} \tag{3}$$

Assume unit depth. While the incline is constant and the vertical component of flow is zero (as given here) the height h is constant. Thus, pressure is constant in x, and the flow is controlled by gravity and shear. The components of the gravity force per unit volume $\chi_x = \gamma(h - y)\tan\theta$ and $\chi_y = -\gamma(h - y)$.

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu}(-\gamma y + \gamma h)\tan\theta \quad \frac{\partial u}{\partial y} = \frac{1}{\mu}(-\gamma \frac{y^2}{2} + \gamma h y + C)\tan\theta \tag{4}$$

$$u = \frac{1}{\mu}(-\gamma \frac{y^3}{6} + \gamma h \frac{y^2}{2} + Cy + D)\tan\theta \tag{5}$$

At $y = 0$, $u = 0$:

$$u(0) = D = 0 \tag{6}$$

At $y = h$, $u,_y = 0$:

$$u,_y(h) = \frac{1}{\mu}\tan\theta \left(\gamma \frac{h^2}{2} + C\right) = 0 \quad \text{thus} \quad C = -\gamma \frac{h^2}{2} \tag{7}$$

Thus:

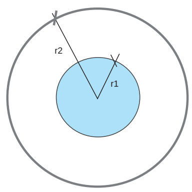
$$u = \frac{\gamma}{\mu}\tan\theta \frac{y}{2}\left(-\frac{y^2}{3} + hy - h^2\right) \tag{8}$$

Also

$$\frac{\gamma \tan\theta}{2\mu}\int_0^h (-\frac{y^3}{3} + hy^2 - h^2 y)\mathrm{d}y = -\frac{\gamma \tan\theta}{8\mu}h^4 = Q \tag{9}$$

h is a function of Q and u then is a function of y, Q, γ, μ and θ. Coordinate system may be different than for given u value.

2. **The Navier-Stokes equations in cylindrical coordinates is given by:**



$$\begin{aligned}
\frac{\mathrm{D}v_r}{\mathrm{D}t} - \frac{v_\theta^2}{t} &= \chi_R - \frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\nabla^2 v_r - \frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right) \\
\frac{\mathrm{D}v_\theta}{\mathrm{D}t} + \frac{v_r v_\theta}{r} &= \chi_b - \frac{1}{r\rho}\frac{\partial p}{\partial \theta} + \nu\left(\nabla^2 v_\theta + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r^2}\right) \\
\frac{\mathrm{D}v_z}{\mathrm{D}t} &= \chi_z - \frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\nabla^2 v_z \qquad (10)
\end{aligned}$$

For flow between two concentric cylinders of fadii r_1 and r_2 where $r_1 < r_2$ in which the inner cylinder is rotating with a constant angular velocity Ω find the velocity profile.
Shear Flow.

$$u_\theta = \frac{\Omega}{(r_2 - r_1)}(r_2 - r), \quad u_r = 0, \quad u_z = 0 \qquad (11)$$

What would the profile be if instead the outside cylinder is rotated.
Again Shear flow.

$$u_\theta = \frac{\Omega}{(r_2 - r_1)}(r - r_1), \quad u_r = 0, \quad u_z = 0 \qquad (12)$$

If the inner cylinder is moving in the axial directin with velocity U_0 (no rotation).

$$u_\theta = 0, \quad u_r = 0, \quad u_z = \frac{U_0}{(r_2 - r_1)}(r_2 - r) \qquad (13)$$

If the flow is subjected only to a pressure gradient $-\frac{\partial p}{\partial x} = k =$ constant.
Pressure driven flow.

$$u_\theta = 0, \quad u_r = 0, \quad u_z = -\frac{k}{2\rho\nu}r^2 + C_1(r) + C_2 \qquad (14)$$

$$\begin{aligned}
u_z(r_1) &= -\frac{k}{2\rho\nu}r_1^2 + C_1(r_1) + C_2 = 0 \\
u_z(r_2) &= -\frac{k}{2\rho\nu}r_2^2 + C_1(r_2) + C_2 = 0 \qquad (15)
\end{aligned}$$

$$u_z(r) = -\frac{k}{2\rho\nu}(r - r_1)(r - r_2) \qquad (16)$$

E6100
Homework 8 **and** 9
Elisabeth Malsch
11/10/2000

Homework 8:

1. **Find the equations of the vortex lines for the flow given by:**

$$u = Az - By \quad v = Bx - Cz \quad w = Cy - Ax. \tag{1}$$

The equation of the vortex is:

$$\nabla\times\vec{u} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial x}\right)\vec{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\vec{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k} = 2C\vec{i} + 2A\vec{j} + 2B\vec{k} \tag{2}$$

The vortex line is parallel to the vorticity:

$$\frac{\xi}{\mathrm{d}x} = \frac{\eta}{\mathrm{d}y} = \frac{\zeta}{\mathrm{d}z} = \frac{1}{\mathrm{d}s} \tag{3}$$

The vortex line parametrized by S is :

$$\{2CS + \xi_0, 2AS + \eta_0, 2BS + \zeta_0\} \tag{4}$$

2. **The components of vorticity in** 3 **dimensions are** ξ, η, ζ **and** $2D$ **flow is given by:**

$$u = \frac{\partial \Psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \Psi}{\partial x} \tag{5}$$

Now the ζ component of vorticity is:

$$\zeta = -\nabla^2\Psi = -\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\Psi\right) + \frac{1}{r^2}\frac{\partial^2\Psi}{\partial\theta^2} = \Psi'' + \frac{1}{r}\Psi' \tag{6}$$

If the vorticity component is represented by the dirac delta function. Then governing equation of the stream function at every point away from the origin is

$$\Psi'' + \frac{1}{r}\Psi'. \tag{7}$$

3. **Show that for a barotropic, inviscid flow, the Navier-Stokes equation can be represented as**

$$\frac{\partial \vec{v}}{\partial t} - \vec{v}\times\vec{\Omega} = -\nabla\left(\frac{1}{2}V^2 + \int\frac{\mathrm{d}P}{\partial\rho} + F_V\right) \tag{8}$$

Recall the following tensor relation:

$$\vec{v}\times\nabla\times\vec{v} = \frac{1}{2}\nabla\left(\vec{v}\cdot\vec{v}\right) - \vec{v}\cdot\nabla\vec{v}. \tag{9}$$

Now the Navier Stokke's equation, with a conservative force and barotropic flow becomes as above.

Homework 9:

1. **The two dimensional Couette flow between parallel plates has an associated stream function Ψ** The stream functions constant, since Couette flow is a shear driven flow and inviscid flow does not shear. Similarly the velocity potential is also zero, there is no energy in the system, no work is done.

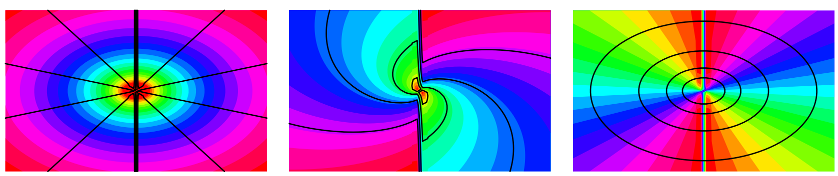
2. **For potential flow described by the Stream function $\Psi = C_1\theta + C_2 \ln r$ where C_1 and C_2 are constants. The corresponding potential function is derived as follows:**

$$\frac{1}{r}\frac{\partial\Psi}{\partial\theta} = \frac{C_1}{r} = \frac{\partial\Phi}{\partial r} \quad \text{thus} \quad \Phi = C_1 \ln r + f(\theta) \tag{10}$$

Also,

$$\frac{1}{r}\frac{\partial\Phi}{\partial\theta} = f'(\theta) = \frac{\partial\Psi}{\partial r} = \frac{C_2}{r} \quad \text{now} \quad \Phi = C_1 \ln r + C_2\theta \tag{11}$$

The flow depends on the constants if $C_1 > C_2$ the flow is circular and the potential varies angularly. If $C_1 = C_2$ the flow is in a skewed figure eight, potential lines and streamlines are parallel. If $C_1 < C_2$ the flow is towards the origin on one axis and away from the origin on the other.



3. **A half body in two-dimensions is generated by a combination of source and uniform stream $\Psi = Ur\sin\theta + m\theta$.**

The stagnation point occurs where the velocities are zero, the velocities are:

$$V_r = \frac{1}{r^2\sin\theta}\frac{\partial\Psi}{\partial\theta} = \frac{Ur\cos\theta + m}{r^2\sin\theta} \quad \text{and} \quad V_\theta = -\frac{1}{r\sin\theta}\frac{\partial\Psi}{\partial r} = -\frac{U}{r}. \tag{12}$$

When V_r and V_θ are zero:

$$Ur\cos\theta = -m \quad \text{and} \quad V_\theta \neq 0 \text{ unless } U = 0 \text{ or } r \to \infty \tag{13}$$

Thus, at the stagnation point the surface is normal to $\vec{\theta}$ and $r = -\frac{m}{U\cos\theta}$. The stagnation point is most likely on the line of symmetry where $\theta = \pi$ and thus $r = \frac{m}{U}$. To find the streamline closest to the surface find the constant for these values:

$$\Psi_0 = m\pi = Ur\sin\theta + m\theta \quad \text{and} \quad r = m\frac{\pi - \theta}{U\sin\theta} \tag{14}$$

The separation D as $\theta \to 2\pi$ and $r \to \infty$ is :

$$m\pi = UD + m2\pi \quad \text{and} \quad D = -\frac{m\pi}{U}. \tag{15}$$

E6100
Homework 11
Elisabeth Malsch
11/26/2000

1. **For slow unsteady motion of an incompressible** $\nabla^2(P+\rho\Omega)=0$**.**

   Assume (1) forces can be written in potential form and (2) the flow is incompressible:
$$\rho\frac{D\vec{u}}{Dt}=\nabla\cdot(P+\rho\Omega)+\mu\nabla^2\vec{u} \tag{1}$$
   Non-dimensionalize the equation.
$$u^*=\frac{u}{U}\quad \nabla^*=\nabla L\quad t^*=t\frac{U}{L}\quad P^*=P\frac{L}{\mu U}\quad (\rho\Omega)^*=(\rho\Omega)^*\frac{\mu U}{L} \tag{2}$$
   Thus:
$$\frac{LU}{\nu}\left(\frac{D\vec{u}}{Dt}\right)^*=(\nabla(P+\rho\vec{u})+\nabla^2u)^* \tag{3}$$
   For a very slow flow.
$$(\nabla(P+\rho\vec{u})+\nabla^2u)^*\approx 0 \quad\text{and}\quad \nabla(P+\rho\vec{u})+\nabla^2u\approx 0 \tag{4}$$
   Using incompressibility
$$\nabla\cdot\nabla(P+\rho\mu)+\nabla^2(\nabla\cdot\vec{u})=0 \quad\text{thus}\quad \nabla^2(P+\rho\vec{u})=0 \tag{5}$$

2. **Velocity, pressure and shear distributions at typical sections of a sphere**

   From class notes. For creeping flows:
$$u=U\cos\theta-2\left(\frac{A}{R^3}+\frac{B}{R}\right)\cos\theta \quad\text{and}\quad v=-U\sin\theta-\frac{A}{R^3}-\frac{B}{R}\sin\theta \tag{6}$$
$$F=2\pi a^2\int_0^{2\pi}\tau_z\sin\theta d\theta \quad\text{and}\quad A=-\frac{Ua^3}{4},\quad B=\frac{3}{4}Ua \tag{7}$$
   From potential flow: For a source at $z=a$, $\Phi=m\ln r$ and $\Psi=m\theta$

3. **Plot the dimensionless pressure** $P\frac{b_1^2}{\mu U_0 L}$ **as the function** x/L **for several values of** $\frac{b_2}{b_1}$**, where the parabolic variation of the gap is given by** $b=b_2+(b_1-b_2)(1-\frac{x}{L})^2$
$$\frac{z}{2\mu}\frac{\partial P}{\partial x}(z-b)+U(1-\frac{z}{b})=u \tag{8}$$

Flow about an Axisymmetric Rankine Body

December 28, 2000

Flow About an axisymmetric Rankine body is generated by a source and a sink placed in a uniform stream of velocity:

$$U = 1\frac{m}{s} \tag{1}$$

The source and sink are located respectively at $x = -1m$, $x = 1m$ on the axis of revolution and are of equal strength, $m = 10\frac{m^3}{s}$:

$$v_r = \frac{m}{4\pi R^2} \tag{2}$$

The assumptions required for the stream an potential functions include that the fluid is incompressible, inviscid and irrotational.

1 Rankine Body Outline and Plot of Some Characteristic Streamlines

The stream function is the summation of the stream functions of the point sources and the far field velocity U.

1.1 Stream and Potential Functions

1.1.1 Point Source

For a point source, the velocity contribution in spherical coordinates is:

$$\{v_r, v_\theta, v_\rho\} := \{\frac{m}{4\pi R^2}, 0, 0\} \tag{3}$$

Solving for the potential:

$$\begin{array}{ll} \dfrac{\partial\Phi}{\partial R} = v_r & \Phi = \displaystyle\int v_r \mathrm{d}R + g(\theta) = \frac{-m}{4\pi R} + g(\theta) \\ \dfrac{1}{r}\dfrac{\partial\Phi}{\partial\theta} = v_\theta & \dfrac{1}{R}\dfrac{\partial}{\partial\theta}(-\dfrac{m}{2\pi R} + g(\theta)) = \dfrac{g'(\theta)}{R} = 0 \end{array} \tag{4}$$

Thus $g(\theta) = Constant$. For convenience, let the arbitrary constant equal zero.

$$\Phi' = -\frac{m}{4\pi R} \tag{5}$$

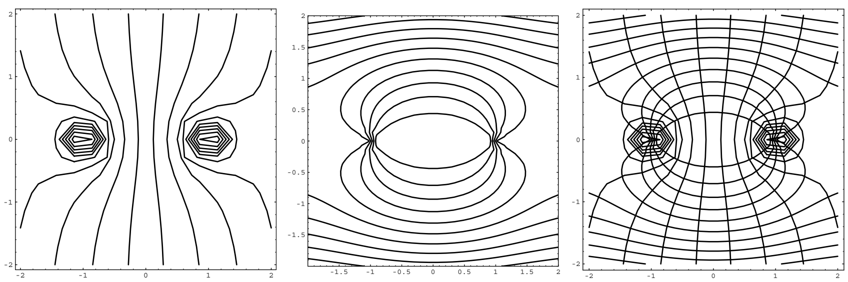


Figure 1: Potential, stream and combined contours respectively

Solving for the stream function:

$$-\frac{1}{R\sin\theta}\frac{\partial\Psi}{\partial R} = v_\theta \qquad \Psi = \int -R\sin\theta v_\theta \mathrm{d}R + h(\theta) = h(\theta)$$
$$\frac{1}{R^2\sin\theta}\frac{\partial\Psi}{\partial\theta} = v_r \qquad \frac{1}{R^2\sin\theta}\frac{\partial h(\theta)}{\partial\theta} = \frac{h'(\theta)}{R^2\sin\theta} = \frac{m}{4\pi R^2} \tag{6}$$

Solving for $h(\theta)$ and setting the remaining constant to zero for convenience.

$$\Psi' = -\frac{m}{4\pi}\cos\theta \tag{7}$$

1.1.2 Uniform Stream on an Axisymmetric Domain

$$\Phi'' = Ux \quad \text{and} \quad \Psi'' = \frac{1}{2}Ur^2 \tag{8}$$

Here r is the distance from the $x-$axis. In spherical coordinates $r = R\cos\theta$ and in cartesian coordinates $r^2 = y^2 + z^2$.

1.1.3 Sums Combine to Form Potential and Stream Functions

The potential (figure:1) in cartesian coordinates is:

$$\Phi = -\frac{m}{4\pi\sqrt{(x+a)^2 + y^2 + z^2}} + \frac{m}{4\pi\sqrt{(x-a)^2 + y^2 + z^2}} + Ux. \tag{9}$$

The stream function (figure:1) in spherical coordinates is:

$$\Psi = -\frac{m}{4\pi}\cos\theta_1 + \frac{m}{4\pi}\cos\theta_2 + \frac{1}{2}UR\sin\theta. \tag{10}$$

The local angles θ_1 and θ_2 in terms of (R, θ) and the distance a from the origin.

$$\tan\theta_1 = \frac{R\sin(\pi-\theta)}{a - R\cos(\pi-\theta)} \quad \text{and} \quad \cos\theta_1 = \frac{a + R\cos\theta}{\sqrt{a^2 + R^2 + 2aR\cos\theta}} \tag{11}$$

Also.

$$\tan(\pi - \theta_2) = -\tan(\theta_2) = -\frac{R\sin(\pi-\theta)}{a + R\cos(\pi-\theta)} \tag{12}$$

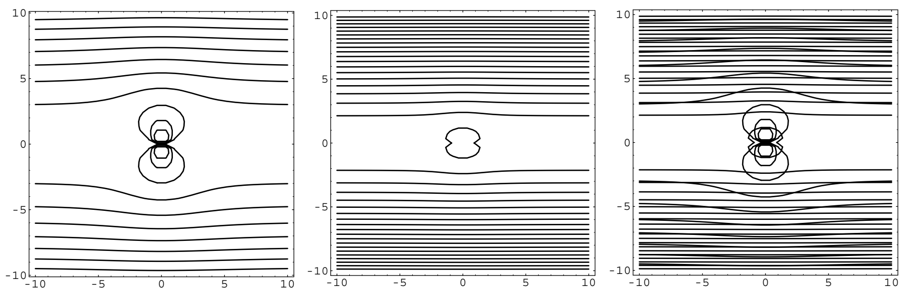


Figure 2: Comparison of doublet and calculated stream function respectively

$$\cos(\pi - \theta_2) = -\cos\theta_2 = \frac{-(a - R\cos\theta)}{\sqrt{a^2 + R^2 - 2aR\cos\theta}} \tag{13}$$

Using these results the stream function can be reconstructed with the global angle.

$$\Psi = -\frac{m}{4\pi}\left(\frac{a + R\cos\theta}{\sqrt{a^2 + R^2 + 2aR\cos\theta}} + \frac{(a - R\cos\theta)}{\sqrt{a^2 + R^2 - 2aR\cos\theta}}\right) + \frac{1}{2}UR^2\sin^2\theta \tag{14}$$

To rewrite in cartesian coordinates recall that $x = R\cos\theta$, $\sqrt{y^2 + z^2} = R\sin\theta$ and $R^2 = x^2 + y^2 + z^2$. Since the problem is axisymmetric it is convenient to write the stream function in cylindrical coordinates where $r^2 = y^2 + z^2$.

$$\Psi = -\frac{m}{4\pi}\left(\frac{a + x}{\sqrt{a^2 + x^2 + r^2 + 2ax}} + \frac{a - x}{\sqrt{a^2 + x^2 + r^2 - 2ax}}\right) + \frac{1}{2}Ur^2 \tag{15}$$

Notice that the contours of the potential and stream functions are orthogonal (figure:1).

Also, if the source and sink had been approximated by a doublet then the streamlines would only have been equal in the far field (figure:2).

1.1.4 Velocity Field

The velocities can be found from either the potential or the stream function (figure:3).

$$\begin{aligned}
u &= 1 - \frac{5\,(-1+x)}{2\,\pi\left((-1+x)^2 + y^2 + z^2\right)^{\frac{3}{2}}} + \frac{5\,(1+x)}{2\,\pi\left((1+x)^2 + y^2 + z^2\right)^{\frac{3}{2}}} \\
v &= \frac{-5\,y}{2\,\pi\left((-1+x)^2 + y^2 + z^2\right)^{\frac{3}{2}}} + \frac{5\,y}{2\,\pi\left((1+x)^2 + y^2 + z^2\right)^{\frac{3}{2}}} \\
w &= \frac{-5\,z}{2\,\pi\left((-1+x)^2 + y^2 + z^2\right)^{\frac{3}{2}}} + \frac{5\,z}{2\,\pi\left((1+x)^2 + y^2 + z^2\right)^{\frac{3}{2}}}
\end{aligned} \tag{16}$$

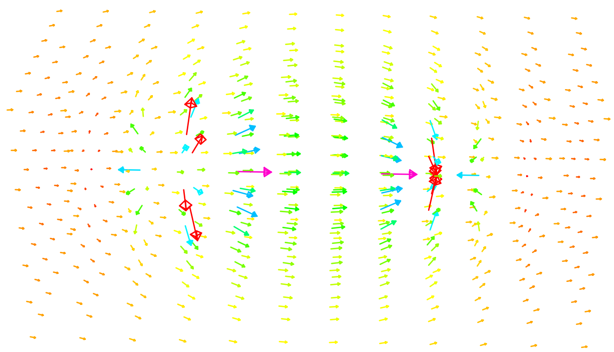


Figure 3: Velocity Field

1.2 Rankine Body

1.2.1 Stagnation Point

The stagnation point occurs when the far field velocity and the radial velocity of the point source at the axis of symmetry.

$$U = \frac{m}{4\pi R_1^2} \quad \text{and} \quad R_1 = \sqrt{\frac{m}{4\pi U}}, \quad R = a + R = a + \sqrt{\frac{m}{4\pi U}} \tag{17}$$

Also $\theta = -\pi$ and $R = r$.

1.2.2 Body Outline

The value of the streamline at the stagnation point can be calculated:

$$\Psi = \frac{-5\left(-1 + \sqrt{\frac{5}{5+8\,\pi+4\,\sqrt{10\,\pi}}} + 2\,\sqrt{\frac{2\,\pi}{5+8\,\pi+4\,\sqrt{10\,\pi}}}\right)}{2\,\pi} = 0 \tag{18}$$

Thus, the equation of the Rankine Body is:

$$\Psi = -\frac{m}{4\pi}\left(\frac{a+x}{\sqrt{(a+x)^2+r^2}} + \frac{a-x}{\sqrt{(a-x)^2+r^2}}\right) + \frac{1}{2}Ur^2 = 0. \tag{19}$$

The equation is not easily solvable for one variable so the plot (figure:4) is made by solving for the distance from the $x-$axis (r) for a given value of x.

From the plot (figure:4) body appears to be a elliptical surface of revolution.

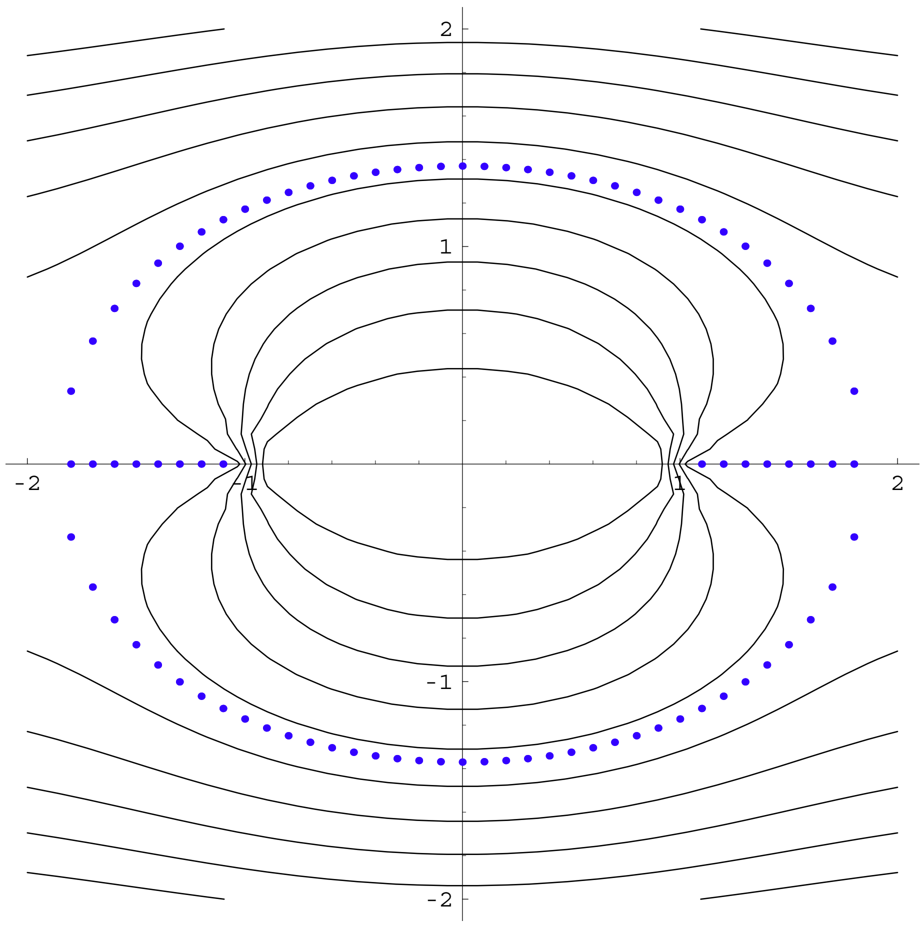


Figure 4: Streamlines and the Rankine Body

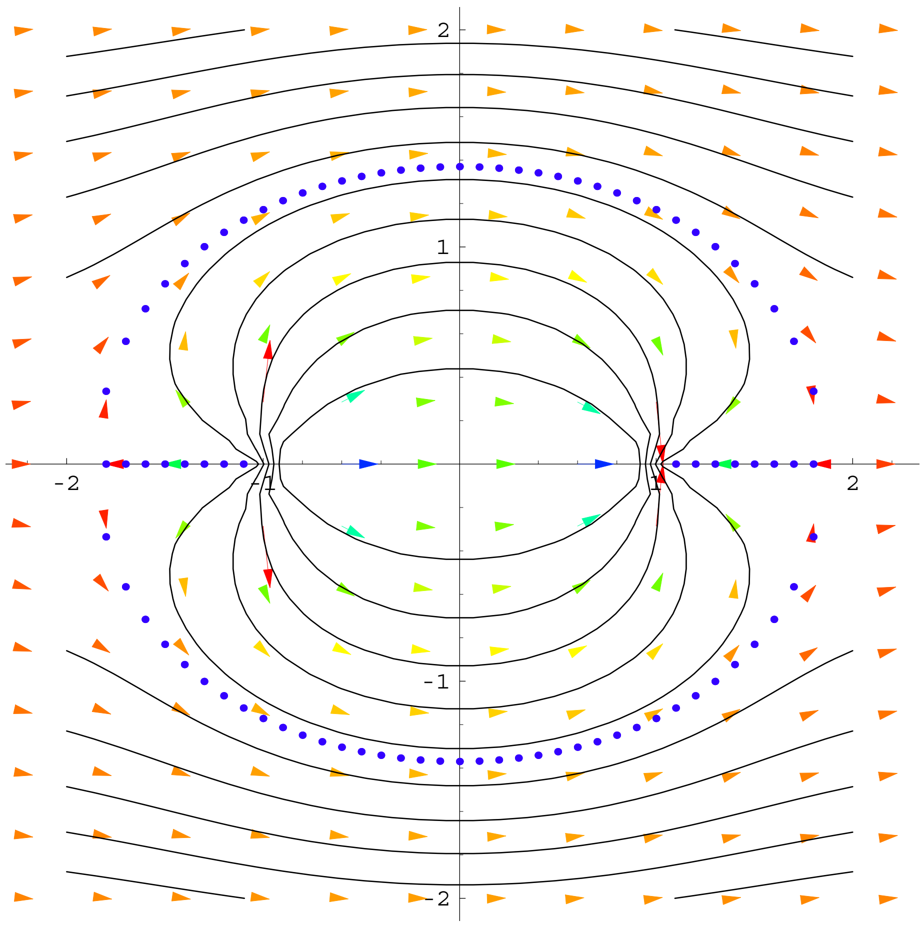


Figure 5: Streamlines and the Rankine Body with velocity field

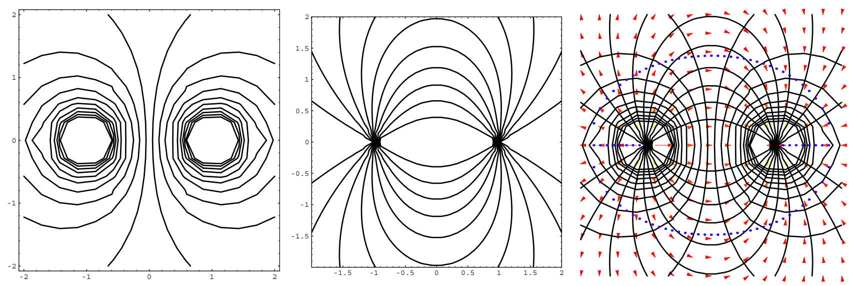


Figure 6: Potential, stream and combined contour with velocity respectively.

2 Unsteady Flow Created from a Moving Source and Sink With No Velocity in the Far Field

If the source and sink are moved by at $-U$. Then the flow pattern is the same as for an ellipsoid, the shape of the rankine body, moving in a fluid which is otherwise at rest. The potential and stream functions are now only the sum of the source contributions and not the far field velocity see figure 6.

$$\Psi = -\frac{m}{4\pi}\left(\frac{a+x}{\sqrt{a^2+x^2+r^2+2ax}} + \frac{a-x}{\sqrt{a^2+x^2+r^2-2ax}}\right) \tag{20}$$

$$\Phi = -\frac{m}{4\pi\sqrt{(x+a)^2+r^2}} + \frac{m}{4\pi\sqrt{(x-a)^2+r^2}} \tag{21}$$

where r is the distance from the $x-$ axis.

The location of the source and sink are a function of time. $a = 1m$, $U = 1m/s$, $m = 10m^3/s$ and $x \to x+t$. The velocity field can be found by taking the derivative of the moving potential. Inverting the velocity equation to find a closed form expression for the pathlines is not trivial.

$$\begin{aligned} x'(t) &= \frac{-5\,(-1+t+x)}{2\,\pi\left(r^2+(-1+t+x)^2\right)^{\frac{3}{2}}} + \frac{5\,(1+t+x)}{2\,\pi\left(r^2+(1+t+x)^2\right)^{\frac{3}{2}}} \\ r'(t) &= \frac{-5\,r}{2\,\pi\left(r^2+(-1+t+x)^2\right)^{\frac{3}{2}}} + \frac{5\,r}{2\,\pi\left(r^2+(1+t+x)^2\right)^{\frac{3}{2}}} \end{aligned} \tag{22}$$

Thus, the pathlines are calculated numerically and by sketch (see figure 8 and attached sketch). Notice that the hand sketch and computer sketch are not quite of the same form especially for the particle initially located at M_1. This is evidence of the sensitivity of the pathline calculation in an unsteady fluid especially close to the source and sink. The Mathematica code used to calculate the pathline numerically is:

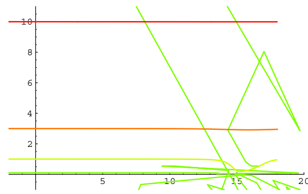


Figure 7: Streaklines from points M_1 (green), M_2 (yellow), M_3 (orange) and M_4 (red) calculated numerically for steady flow.

```
pathLine[m_, h_] :=
  Module[{pos = m},
    Prepend[Table[ pos += (xrvel[pos, t + h]*h), {t, 1, 20, h}], m]]
```

Where m is the position of the particle at time $t = 0$ and h is the time step `xvel[·]` is the function which returns velocity in terms of position and time. Here the derivative is calculated numerically using Euler's method, which is a linear interpolation based on the first two terms of a taylor series expansion.

The streaklines can also be calculated numerically. The Mathematica code used to calculate the position of a particle which passed the marker position m at a time $t < 20s$ at time $t = 20s$:

```
ppM[m_, t_, h_] :=
  Module[{pos = m, t1 = 20, t2 = t},
  {While[t2 < t1 , pos += (xrvel[pos, t2]*h); t2 += h], pos}[[-1]]]
```

Numerical calculations of the streaklines are not stable near the sink figure 9. The streaklines should never cross the $x-$axis, since the streamlines do not cross the $x-$axis at anytime t see figure 6. Notice that the numerical calculations for steady flow also exhibit instability about the sink. see figure 7.

The values for the point M_1 from the numerical calculation is quite unreliable, refer instead to the hand sketch. Better results could be obtained with a better method for finding the derivative, like a two dimensional Runge-Kutta method.

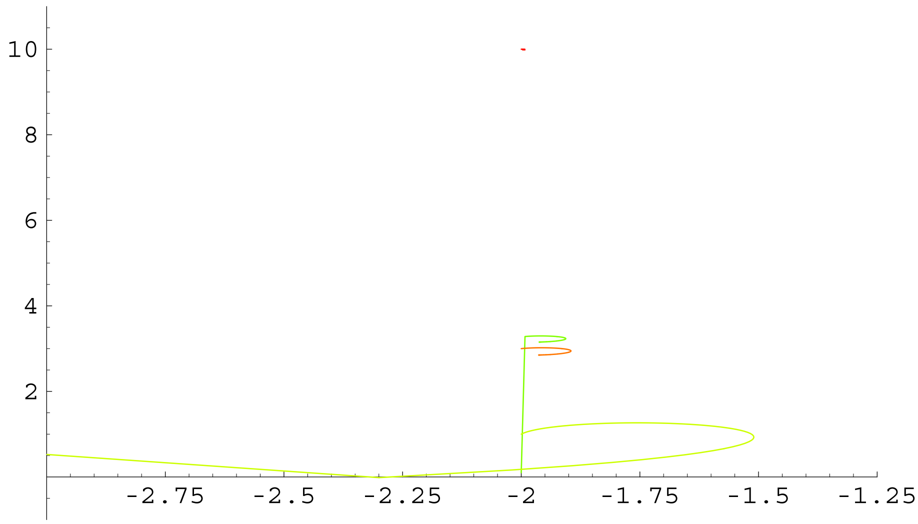


Figure 8: Pathlines of particles which were located at M_1(green), M_2(yellow), M_3 (orange) and M_4 (red) at time $t = 0$

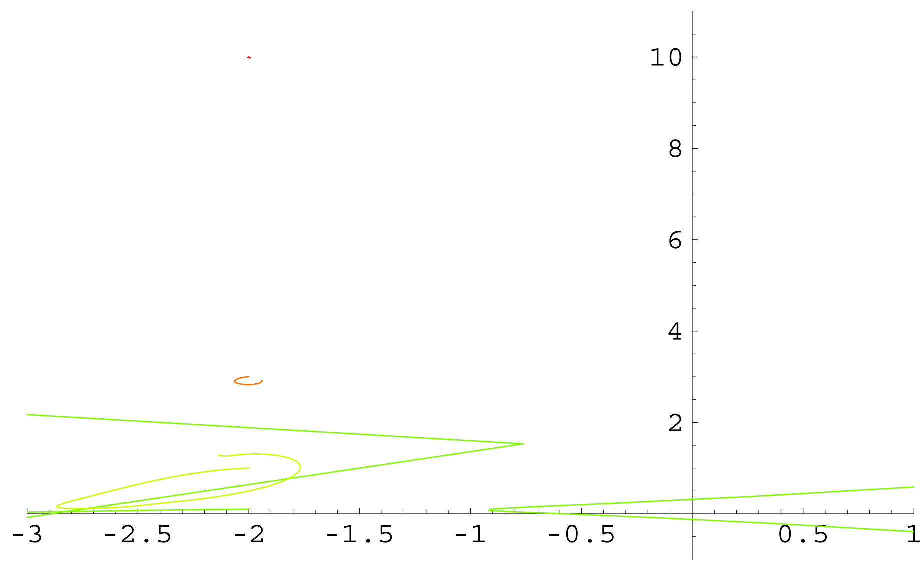


Figure 9: Streaklines from points M_1 (green), M_2 (yellow), M_3 (orange) and M_4 (red) calculated numerically for unsteady flow.